ECTURE 14 : Semidirect Product by an abelian group

Suppose That G is The semidirect product of the subgroups A and H ($G = A \circledS H$) and that A is abelian and normal. In This lecture we show That every representation of G can be constructed ~~by~~ from those of certain subgroups of H. This is The method of "little groups" of Wigner and Mackey.

We will apply this method To find The irreducible representations of The Weyl groups of Type B_n, C_n and D_n.

(1) THE MAIN THEOREM

Suppose That $G = A \circledS H$ is the semidirect product of A and H, and that the subgroup A is abelian and normal.

By definition of semidirect product, $AH = G$ and $A \cap H = \{e\}$. Hence every element of G can be written uniquely as a product of an element of A and an element of H.

By hypothesis A is abelian, so every irreducible representation of A is one-dimensional. Write $\mathbb{X}$ for the set of inequivalent representations.

[Each χ in $\mathbb{X}$ is a group homomorphism from G To $\mathbb{C}^*$].

Because A is normal, the group G acts on $\mathbb{X}$: for all $s \in G$, define:

$$(s \cdot \chi)(a) \equiv \chi(\underbrace{s^{-1} a s}_{\text{in } A}), \ \forall a \in A.$$

This action is well defined because

$$(s_1 s_2 \cdot \chi)(a) = \chi(s_2^{-1} s_1^{-1} a s_1 s_2) = (s_2 \cdot \chi)(s_1^{-1} a s_1) = s_1 \cdot (s_2 \cdot \chi)(a) \quad [\forall a \in A]$$

$$= s_1 \cdot (s_2 \cdot \chi) \quad \forall \chi \in \mathbb{X} \ \forall s_1, s_2 \in G.$$

Choose representatives $x_1, \dots, x_r$ for the orbits of G on $\overline{X}$:

$$\overline{X} = \biguplus_{i=1}^{r} G \cdot x_i ,$$

and set

$$H_i = \{ h \in H : h \cdot x_i = x_i \}$$

$\forall i = 1..r$. Notice that H_i is a subgroup of H, and it's equal to the stabilizer in H of x_i.

For each $i = 1..r$, and each representation ρ of H_i, we can define a representation $\theta_{i,\rho}$ of G as follows: consider the subgroup $G_i = A \circledS H_i \leq A \circledS H = G$ of G, and the representation $\tilde{x}_i \otimes \tilde{\rho}$ of G_i defined by:

$$(\tilde{x}_i \otimes \tilde{\rho})(a h_i) = x_i(a)\, \rho(h_i).$$

then we set $\boxed{\theta_{i,\rho} = \mathrm{Ind}_{G_i}^{G} (\tilde{x}_i \otimes \tilde{\rho})}$.

<u>Remark</u> - $\tilde{x}_i \otimes \tilde{\rho}$ is a well defined representation of $G_i = A \circledS H_i$ because

$$\tilde{x}_i \otimes \tilde{\rho}\, (a h_i \tilde{a} \tilde{h}_i) = \tilde{x}_i \otimes \tilde{\rho}\, (\underbrace{a h_i \tilde{a} h_i^{-1}}_{\text{in } A} \underbrace{h_i \tilde{h}_i}_{\text{in } H}) =$$

$$= x_i(\underbrace{a}_{\text{in } A} \underbrace{h_i \tilde{a} h_i^{-1}}_{\text{in } A})\, \rho(h_i \tilde{h}_i) =$$

$$= x_i(a)\, x_i(h_i \tilde{a} h_i^{-1})\, \rho(h_i)\, \rho(\tilde{h}_i) =$$

$$= x_i(a)\, (h_i^{-1} \cdot x_i)(\tilde{a})\, \rho(h_i)\, \rho(\tilde{h}_i) =$$

$$\underset{\substack{\uparrow \\ h_i \in H_i = \text{the stabilizer} \\ \text{of } x_i \text{ inside } H}}{=} x_i(a)\, x_i(\tilde{a})\, \rho(h_i)\, \rho(\tilde{h}_i) \underset{\substack{\uparrow \\ x_i(\tilde{a}) \in \mathbb{C}^*}}{=} [x_i(a)\, \rho(h_i)][x_i(\tilde{a})\, \rho(\tilde{h}_i)] =$$

$$= (\tilde{x}_i \otimes \tilde{\rho})(a h_i)\, (\tilde{x}_i \otimes \tilde{\rho})(\tilde{a} \tilde{h}_i)$$

$\forall a, \tilde{a} \in A$, $\forall h_i, \tilde{h}_i \in H_i$. $\square$

The representations $\{\Theta_{i,\rho} : i=1..r, \rho \text{ irred. repr. of } H_i\}$ are irreducible, and exhaust the set of irreducible inequivalent representations of G. Indeed, the following theorem holds:

HEOREM: ① $\Theta_{i,\rho}$ is irreducible

② $\Theta_{i,\rho} \sim \Theta_{i',\rho'}$ if and only if $i=i'$ and $\rho\sim\rho'$

③ Every irreducible representation of G is isomorphic to one of the $\Theta_{i,\rho}$'s.

Thus we obtain a complete classification of the irreducible representations of $G = A \circledS H$.

PROOF: ① Because the representation $\Theta_{i,\rho}$ of G is induced from the representation $\tilde{\chi}_i \otimes \tilde{\rho}$ of G_i, we can use Mackey's irreducibility criterion for irreducibility.

> Recall this criterion: If $K \leq G$ and $\Theta = \mathrm{Ind}_K^G \varphi$ then Θ is irreducible if and only if the representations $\mathrm{Res}_{K_s} \varphi$ and φ^s of K_s are disjoint, $\forall s \in G-K$.
> We have set: $K_s = (sKs^{-1}) \cap K$ and
> $\varphi^s(x) = \varphi(s^{-1}xs)$, $\forall x \in K_s$.

For all s in $G - G_i$, set $K_s = (sG_is^{-1}) \cap G_i$. For brevity of notations set $\tilde{\chi}\otimes\tilde{\rho} = \varphi$. Then $\forall x \in K_s$

$$(\mathrm{Res}_{K_s}\varphi)(x) = \varphi(x) \; ; \qquad \varphi^s(x) = \varphi(s^{-1}xs).$$

We need to prove that these representations of K_s are disjoint.

<u>Claim</u>: it's enough to prove that the restriction to A of these representations of K_s are disjoint.

▸ First of all, notice that A is a subgroup of $[sG_is^{-1} \cap G_i] = K_s$ (because $A \subseteq G_i$ and $sAs^{-1} = A$). If $\mathrm{Res}_{K_s}\psi$ and ψ^s had a common factor ρ', then the restriction to A of ρ' would be a common factor of the restrictions to A of $\mathrm{Res}_{K_s}\psi$ and ψ^s. □

So the plan is to show that $\mathrm{Res}_A(\mathrm{Res}_{K_s}\psi)$ and $\mathrm{Res}_A(\psi^s)$ are disjoint.

We notice that:

- $\mathrm{Res}_A(\psi^s)(a) = \psi^s(a) = \psi(\underbrace{s^{-1}as}_{\text{in } A}) = \chi(s^{-1}as)\,\rho(1_G) =$
  $= (s\cdot\chi)(a)\,\rho(1_G) = (s\cdot\chi)(a)\,\mathbb{1}_{\dim\rho} \quad \forall a\in A$

$\Longrightarrow \mathrm{Res}_A(\psi^s) = (s\cdot\chi)^{\oplus\dim\rho}$.

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$\Longrightarrow \mathrm{Res}_A(\psi^s) = (\chi)^{\oplus\dim\rho}$.

Therefore, we just need to show that $s\cdot\chi$ and χ are not isomorphic $\forall s\in G-G_i$.

[They are 1-dimensional representations of A, so not isomorphic simply means different...]

● It is of course enough to prove that the stabilizer of χ_i in G is equal to G_i, and this is easy to do:

$G_i \subseteq St_G(\chi_i)$ because

$$\underbrace{(ah_i)}_{\text{in } G_i} \cdot \chi_i(\tilde{a}) = \chi_i(h_i^{-1} \underbrace{a^{-1}\tilde{a}\,a}_{\substack{=\tilde{a} \text{ because} \\ A \text{ is abelian}}} h_i) = \chi_i(h_i^{-1}\tilde{a}h_i) =$$

$$= (h_i \cdot \chi_i)(\tilde{a}) \underset{\substack{\uparrow \\ h_i \in H_i = St_H(\chi_i)}}{=} \chi_i(\tilde{a}) \quad \forall \tilde{a} \in A$$

$$\Rightarrow ah_i \cdot \chi_i = \chi_i, \quad \forall ah_i \in G_i \Rightarrow G_i \subseteq St_G(\chi_i).$$

Viceversa, assume that $s = ah$ belongs to $St_G(\chi_i)$.

Then

$$\chi_i(\tilde{a}) = (s \cdot \chi_i)(\tilde{a}) \underset{\substack{\uparrow \\ s = ah}}{=} \chi_i(h^{-1}\underbrace{a^{-1}\tilde{a}a}_{=\tilde{a}}h) = (h \cdot \chi_i)(\tilde{a})$$

so $h \in St_H(\chi_i) = H_i$ and $s = ah \in G_i = A \circledS H_i$.

This proves that $G_i = St_G(\chi_i)$. It follows that $s \cdot \chi_i \neq \chi_i$, $\forall s \in G - G_i$ and that $Res_A(\psi^s)$ and $Res_A(Res_{K_s} \psi)$ are disjoint representations of A.

$\Rightarrow$ ψ^s and $Res_{K_s} \psi$ are disjoint representations of K_s, $\forall s \in G - G_i$.

$$\Rightarrow \Theta_{i,\rho} = Ind_{G_i}^G \psi = Ind_{G_i}^G (\tilde{\chi}_i \otimes \tilde{\rho}) \text{ is irreducible.}$$

$\square$

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② Next, we prove that if $\Theta_{i,\rho} \cong \Theta_{i',\rho'}$ then $i = i'$ and $\rho \cong \rho'$.

To show that $i = i'$, we show that the index i determines the restriction of $\Theta_{i\rho}$ to A (up to isomorphism). So if $\Theta_{i\rho} \cong \Theta_{i'\rho'}$ then $\mathrm{Res}_A(\Theta_{i\rho}) \cong \mathrm{Res}_A(\Theta_{i'\rho'})$ and i must equal i'.

$$\mathrm{Res}_A(\Theta_{i\rho}) = \mathrm{Res}_A\left(\mathrm{Ind}_{G_i}^{G} \psi\right) = \mathrm{Res}_A\Big(\bigoplus_{x \in G/G_i} x.\psi\Big) =$$

$x \in G/G_i$ cosets of G_i in G (not necessarily a group)

$$= \bigoplus_{x \in G/G_i} \mathrm{Res}_A(x.\psi) =$$

$$= \bigoplus_{x \in G/G_i} x.\left(\mathrm{Res}_A \psi\right) = \bigoplus_{x \in G/G_i} x.\left(\chi_i^{\oplus \dim \rho}\right) =$$

$$= \bigoplus_{x \in G/G_i} (x.\chi_i)^{\oplus \dim \rho}.$$

Notice that this is a direct sum of characters of A that lie in the orbit of χ_i.

If $i \neq i'$, then $(G.\chi_i) \cap (G.\chi_{i'}) = \phi$. Indeed the χ_i's have been choosen to be representatives for the orbits of G on X, then $\mathrm{Res}_A(\Theta_{i,\rho})$ and $\mathrm{Res}_A(\Theta_{i',\rho'})$ have no common factors.

We now prove that if $\Theta_{i,\rho} \sim \Theta_{i',\rho'}$ then $\rho \sim \rho'$. By the previous remarks, we can assume that $i = i'$.

Set W_i = isotypic component of χ_i in $\mathrm{Res}_A(\Theta_{i,\rho})$.

[If W is the representation space for $\Theta_{i,\rho}$, then
$W_i = \{ \underline{w} \in W : \Theta_{i,\rho}(a)\underline{w} = \chi_i(a)\underline{w}, \ \forall a \in A \}$]

Because H_i is - by definition - The stabilizer of χ_i (in H)
W_i = The isotypic component of χ_i in $\mathrm{Res}_A(\Theta_{i\rho})$
is stable under H_i: $\Theta_{i\rho}(h_i) W_i \subseteq W_i \quad \forall h_i \in H_i \iff$

$\Rightarrow \Theta_{i\rho}(a)\Theta_{i\rho}(h_i)\underline{w}_i = \chi_i(a)\Theta_{i\rho}(h_i)\underline{w}_i \quad \forall a \in A, \forall h_i \in H_i, \forall \underline{w}_i \in W_i$

We prove this by a direct computation:

$$\Theta_{i\rho}(a)\Theta_{i\rho}(h_i)\underline{w}_i = \Theta_{i\rho}(ah_i)\underline{w}_i = \Theta_{i\rho}(h_i h_i^{-1} a h_i)\underline{w}_i =$$

$$= \Theta_{i\rho}(h_i)\,\Theta_{i\rho}(\underbrace{h_i^{-1}a h_i}_{\text{in } A})\underbrace{\underline{w}_i}_{\substack{\text{in The isotypic}\\ \text{of } \chi_i \text{ in } \mathrm{Res}_A(\Theta_{i\rho})}} = \Theta_{i\rho}(h_i)\,\chi_i(h_i^{-1}ah_i)\underline{w}_i =$$

$$= \Theta_{i\rho}(h_i)(h_i\cdot\chi_i)(a)\underline{w}_i \underset{\substack{h_i \in H_i = \text{stablizer}\\ \text{of } \chi_i \text{ in } H}}{=} \Theta_{i\rho}(h_i)\,\chi_i(a)\underline{w}_i \underset{\chi_i(a) \in \mathbb{C}^*}{=} \chi_i(a)\,\Theta_{i\rho}(h_i)\underline{w}_i \checkmark$$

So we can consider The representation of H_i on W_i -

<u>Claim</u>: H_i acts on W_i by ρ.

<u>proof</u> - Let W be The vector space of $\Theta_{i\rho}$, and let W_φ be the vector space of $\varphi = \tilde{\chi} \otimes \tilde{\rho}$. Then

$$W = \mathrm{Ind}_{G_i}^{G} W_\varphi = \bigoplus_{x \in G/G_i} x W_\varphi = \bigoplus_{h \in H/H_i} h W_\varphi .$$

Notice that

$$\Theta_{i\rho}(a)\, h\underline{w} \underset{\substack{ah = h\,\underbrace{h^{-1}ah}_{\text{in } A \subseteq G_i}}}{=} h \cdot \varphi(h^{-1}ah)\underline{w} \underset{h^{-1}ah \in A}{=} h\cdot\chi_i(h^{-1}ah)\,\underline{w} = h\cdot(h\cdot\chi_i)(a)\underline{w}$$

$\forall \underline{w} \in W_\varphi$

$\Rightarrow$ A acts on $h\cdot W_\varphi$ by $h\cdot\chi_i$

$\Rightarrow$ The isotypic component of χ_i in $\mathrm{Res}_A(\Theta_{i\rho})$ is $e\cdot W_\varphi$.

$\Rightarrow$ any element $x = ah_i \in G_i$ acts on W_i by $\psi = \tilde{\chi}_i \otimes \tilde{\rho}$.

$\Rightarrow$ any element of $H_i \leq G_i$ acts on W_i by ρ $\square$

This proves that $\Theta_{i\rho}$ determines ρ uniquely, as ρ is the representation of H_i on the isotypic component of χ_i in $\mathrm{Res}_A(\Theta_{i\rho})$.

So if $\Theta_{i\rho} \cong \Theta_{i'\rho'}$, then $i = i'$ and $\rho \cong \rho'$.

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(3) Finally, we prove that any irreducible representation of G is isomorphic to one of the $\Theta_{i,\rho}$'s.

Let (σ, U) be an irreducible repr. of G. Look at the decomposition of $\mathrm{Res}_A \sigma$ in isotypic components of irred. representations of A:

$$\mathrm{Res}_A \sigma = \bigoplus_{\chi \in X} U(\chi).$$

$X = \{$all irred. inequiv. repr. of $A\}$

Choose $\chi \in X$ st $U(\chi) \neq \{0\}$, and let i be the index st. $G\cdot\chi = G\chi_i$. [Remember that $\chi_1 \dots \chi_r$ are representatives for the orbits of G on X].

Any elements $s \in G$ acts on $\mathrm{Res}_A \sigma$ by permuting the $U(\chi)$'s (sending $U(\chi)$ into $U(s\cdot\chi)$), because $A \trianglelefteq G$. So there exists an element $\tilde{s} \in G$ st $\tilde{s}\cdot U(\chi) = U(\chi_i)$. As a consequence, $U(\chi_i)$ is also non zero. The group H_i is the isotropy group of χ_i in H, so it acts on $U(\chi_i)$. Let (U_i, ρ) be an irreducible sub-representation of the representation of H_i on $U(\chi_i)$.

We notice that $G_i = A \circledS H_i$ acts on U_i by $\psi = \tilde{\chi}_i \otimes \tilde{\rho}$. So the restriction of (U, σ) to G_i contains ψ, at least once.

It follows by Frobenious reciprocity that (U, σ) contains the induced representation $\mathrm{Ind}_{G_i}^{G} \psi = \theta_{i,\rho}$ at least once. But (U, σ) is irreducible, so we must have $\sigma = \theta_{i,\rho}$.