$$\iff \sum_{s \in H\backslash G/H} m(\rho, \mathrm{Ind}_{H_s}^{H} \rho^s) = 1$$

- $$\underset{\text{Frob rec.}}{\iff} \sum_{s \in H\backslash G/H} m(\mathrm{Res}_{H_s}\rho, \rho^s) = 1.$$

If $s \in H$, then $H_s = sHs^{-1} \cap H = H$, so $\mathrm{Res}_{H_s} \rho = \rho$.
Also, $\rho^s(h) = \rho(s^{-1}hs) \underset{s\in H}{=} \rho(s^{-1})\rho(h)\rho(s) \Rightarrow \rho^s \cong \rho$

$\Rightarrow m(\rho_{H_s}, \rho^s) = 1.$

Therefore, we can say that

- $\mathrm{Ind}_H^G \rho$ irreducible $\iff m(\rho_{H_s}; \rho^s) = 0 \quad \forall s \in G-H$.

Here ρ_{H_s} and ρ^s could be reducible, and $m(\rho_{H_s}, \rho^s) = 0$ exactly when ρ_{H_s} and ρ^s have no common factors, i.e. they are disjoint. ▨

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4. Semidirect product

- We apply Mackey's irreducibility criterion to the study of the representations of semidirect products by an <u>abelian</u> normal subgroup.

Suppose that A and H are subgroups of G st

1) $A \trianglelefteq G$
2) A is abelian
3) $G = A \circledS H$, i.e. every element of G can be written uniquely as a product ah, with $a \in A$, $h \in H$.

[this is equivalent to: $G = AH$ and $A \cap H = \{1\}$].

Because A is assumed to be abelian, every irreducible representation of A is one-dimensional.

Write $X = \mathrm{Hom}(A, \mathbb{C}^*) = \{$ irred. inequivalent representations$\} = \{$irred. inequivalent characters$\}$.

The group G acts on X by:

$$(s \cdot \chi)(a) = \chi(s^{-1} a s). \quad (*)$$

Because $A \trianglelefteq G$, $s \cdot \chi$ is well defined.

[$s \cdot \chi$ is a representation of A, because we can write

$$(s \cdot \chi)(a_1 a_2) = \chi(\underbrace{s^{-1} a_1 s}_{\text{in } A} \underbrace{s^{-1} a_2 s}_{\text{in } A}) = (s \chi)(a_1)(s \chi)(a_2).$$

Moreover $(*)$ gives an action of G on X because

$$(s_1 s_2 \cdot \chi)(a) = \chi((s_1 s_2)^{-1} a (s_1 s_2)) = \chi(s_2^{-1}(s_1^{-1} a s_1) s_2) =$$

$$= (s_2 \cdot \chi)(s_1^{-1} a s_1) = s_1 \cdot (s_2 \cdot \chi)(a) \quad \forall a \in A$$

$$\Rightarrow (s_1 s_2) \cdot \chi = s_1 \cdot (s_2 \cdot \chi) \quad \forall s_1, s_2 \in G.]$$

Write X as a union of G-orbits:

$$X = G\cdot x_1 \uplus G\cdot x_2 \uplus \dots \uplus G\cdot x_r.$$

For all $i = 1..r$, write H_i for the isotropy group of x_i in H

$$H_i = \{ g \in H : g\cdot x_i = x_i \}.$$

Then $H_i \leq H$ and $G_i = A \circledS H_i$ is a subgroup of $G = A \circledS H$.

We can construct representations of G_i as follows:

- For each representation ρ of H_i, consider the composition $\tilde{\rho}$ of ρ with the projection of G_i onto H_i:

$$\tilde{\rho} : G_i \xrightarrow{\pi_i} H_i \xrightarrow{\rho} GL(W_\rho). \quad \left(\begin{array}{l} \text{for } g_i = a h_i, \\ \tilde{\rho}(g_i) = \rho(h_i) \end{array} \right)$$

Then $\tilde{\rho}$ is a well defined repr. of G_i.

[Notice that $A \trianglelefteq G_i$, and $G_i = A \circledS H_i$, so there's a projection of G_i onto $G_i/A \cong H_i$. We have denoted this projection by π_i].

- By construction, x_i is a representation of A. Extend x_i to a representation $\tilde{x}_i$ of $G_i = A \circledS H_i$ by:

$$\tilde{x}_i(a h_i) = x_i(a).$$

[Check that it's a well defined representation:

$$\tilde{x}_i(a h \tilde{a} \tilde{h}) = \tilde{x}_i(a \underbrace{h \tilde{a} h^{-1}} h \tilde{h}) = x_i(a \underbrace{h \tilde{a} h^{-1}}_{\text{in } A}) =$$

$= \chi_i(a)\, \chi_i(h\tilde{a}h^{-1}) = \chi_i(a)\,(h^{-1}\cdot\chi_i)(\tilde{a})$.

Because $h \in H_i$ = isotropy group of χ_i in H, $h^{-1}\cdot\chi_i = \chi_i$.

$\Rightarrow \tilde{\chi}_i(a h \tilde{a} \tilde{h}) = \chi_i(a)\,\chi_i(\tilde{a}) = \tilde{\chi}_i(ah)\,\tilde{\chi}_i(\tilde{a}\tilde{h})$.

$\forall\, a, \tilde{a} \in A,\ \forall\, h, \tilde{h} \in H_i$.]

Consider the tensor product

$$\tilde{\chi}_i \otimes \tilde{\rho}.$$

This is a representation of $G_i = A \rtimes H_i$.

[We have one such representation for each $i = 1 \dots r$ and for each irred. repr. ρ of H_i.]

If we induce the various $\tilde{\chi}_i \otimes \tilde{\rho}$ to G, then we find ALL the inequivalent irreducible representations of G.

Proposition -

(1) $\Theta_{i,\rho} \overset{\text{def}}{=} \mathrm{Ind}_{G_i}^{G}(\tilde{\chi}_i \otimes \rho)$ is irreducible

(2) $\Theta_{i,\rho} \cong \Theta_{i',\rho'}$ iff $i = i'$ and $\rho \cong \rho'$

(3) Every irreducible representation of G is isomorphic to one of the $\Theta_{i,\rho}$'s.

proof: next time!

Examples : once again .. D_n !

Case 1 : n odd

- conjugacy classes $\{e\}$
$\{a^k, a^{-k}\}$ $k = 1 .. \frac{n-1}{2}$
$\{a^j b : j = 0 .. n-1\}$

- $D_n = C_n \circledS \mathbb{Z}_2$

here $A = C_n$ is abelian and normal.

Write $D_n = \langle a, b \mid a^n = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle$,
then $A = \langle a \rangle = C_n$, and $H = \mathbb{Z}_2 = \langle b \rangle$.

- $X = \{$ineq. irred. rep.s of $A\} = \{\chi_j : j = 0 .. n-1\}$
where $\chi_j(a) = \omega^j$, and ω is a fixed primitive n^{th} root of unity.

- Action of G on X :

$$s \cdot \chi_j = \chi_j \quad \forall s \in A$$

so it's enough to look at $b \cdot \chi_j$.
We have :

$$b \cdot \underset{\substack{\uparrow \\ \text{Trivial}}}{\chi_0} = \chi_0$$

but $b \cdot \chi_k = \chi_{n-k}$ $\forall k = 1 .. \frac{n-1}{2}$. Indeed

- $(b \cdot \chi_k)(a) = \chi_k(b^{-1}ab) = \chi_k(a^{-1}) = \omega^{-k} = \chi_{-k}(a)$.

So we can choose

$$\chi_0, \chi_1, \dots, \chi_{\frac{n-1}{2}}$$

as representatives for the orbits of G on $\hat{X}$.

the corresponding isotropy groups in $H = \mathbb{Z}_2 = \langle b \rangle$ are:

$$H_0 = \mathbb{Z}_2 = H;\ H_1 = \{e\};\ H_2 = \{e\};\ \dots;\ H_{\frac{n-1}{2}} = \{e\}.$$

Let's construct the representations $\theta_{i,\rho}$ ....

- If $i = 0$, $H_0 = \mathbb{Z}_2$ has two inequivalent irreducible representations (defined by: $\rho_0^1(b) = 1$; $\rho_0^2(b) = -1$).

Because $H_0 = H$, $G_0 = A \circledS H_0 = A \circledS H = G$.

Notice that

- $\tilde{\chi}_0$ is the trivial representation of $G_0 = G$ ($\tilde{\chi}_0(ah) = \tilde{\chi}_0(a) = 1$ $\forall a \in A$ $\forall h \in H$)
- $\tilde{\rho}_0^1$ is also the trivial representation of $G_0 = G$:

$$\tilde{\rho}_0^1(ah) = \rho_0^1(h) = 1 \quad \forall a \in A, h \in H$$

$\tilde{\chi}_0 \otimes \tilde{\rho}_0^1$ is the trivial representation, and clearly

$$\mathrm{Ind}_{G_0 = G}^{G}(\tilde{\chi}_0 \otimes \tilde{\rho}_0^1) = \tilde{\chi}_0 \otimes \tilde{\rho}_0^1 \text{ is the trivial repr.}$$

- $\tilde{\rho}_0^2$ is a 1-dimensional representation of $G_0 = G$ defined by $\tilde{\rho}_0^2(a^i b^j) = \rho_0^2(b^j) = (-1)^b$.

Clearly, $\mathrm{Ind}_{G_0}^{G}(\tilde{\chi}_0 \otimes \tilde{\rho}_0^2) = \tilde{\chi}_0 \otimes \tilde{\rho}_0^2 = \tilde{\rho}_0^2$. This is the one-dimensional character of D_n defined by

$$\chi(a) = 1; \quad \chi(b) = -1.$$

- If $i = 1 \dots \frac{n-1}{2}$, then $H_i = \{e\}$ and ρ can only be the trivial representation of H_i. We have:

$$\left.\begin{array}{l} \tilde{\chi}_i(ah) = \tilde{\chi}_i(a) = \omega^i \\ \tilde{\rho}(ah) = \rho(h) = 1 \end{array}\right\} \quad \forall a \in A, \forall h = e \in H_i$$

Basically, $G_i = A$ and $\tilde{\chi}_i \otimes \tilde{\rho} = \chi_i$.

the induced representation $\mathrm{Ind}_{G_i}^{G} \tilde{\chi}_i \otimes \tilde{\rho}$ is simply the induced $\mathrm{Ind}_A^G \chi_i$.

This is a 2-dimensional representation, whose restriction to A is $\chi_i \oplus \chi_{n-i}$.

Notice that $a \rightsquigarrow \begin{bmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{bmatrix}$, $b \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

These are <u>all</u> the irreducible inequivalent representations of D_n.