# Lecture 13

## 1. General Remarks

<u>Question</u> - Given an <u>irreducible</u> representation $\Theta$ of $G$, how can we determine whether $\Theta$ is induced by "some" representation $\rho$ of "some" subgroup $H$ of $G$?

The following proposition answers the question when $G$ has a normal subgroup.

<u>Proposition</u> - Let $A$ be a proper <u>normal</u> subgroup of $G$, and let $(\Theta, V)$ be an <u>irreducible</u> representation of $G$. Then

(a) either the restriction to $A$ of $\Theta$ is isotypic,

(b) or there exists a subgroup $H$ of $G$, unequal to $G$ and containing $A$, and an irreducible representation $\rho$ of $H$ such that $\Theta$ is induced by $\rho$.

<u>proof</u> - Look at $\mathrm{Res}_A \Theta$, and decompose $V$ as a direct sum of isotypic components of irred. representations of $A$:

$$V = V_{i_1} \oplus \dots \oplus V_{i_k}.$$

If $k=1$, then $V$ is isotypic and (a) holds. Assume that $k \neq 1$. Notice that $G$ permutes the $V_j$'s

because $H$ is normal, $s \cdot V_j$ is again a representation of $H$: $h \cdot s \cdot v = s \cdot (s^{-1}hs) \cdot v$ of $H$, For each irred rep. $\rho$ of $H$, set $\rho_s : H \to GL(F_\rho)$, $h \mapsto \rho(s^{-1}hs)$. Then if ... $V_j = \dots$ the ... $s \cdot V_j = \dots$

and - because $V$ is irreducible - $G$ permutes them transitively. Set $H =$ isotropy group of $V_{i_1} = \{g \in G : g \cdot V_{i_1} = V_{i_1}\}$. Then $A \leq H \lneq G$.

[It is obvious that $H$ contains $A$. The fact that $H$ is nequal to $G$ follows from the fact that

$$1 < K = \#\text{orbits of } G \text{ on the set } \{V_{i_j}\}_{j=1..k} = \frac{|G|}{|H|}$$ $\rightarrow$ isotropy group of $V_{i_1}$. ]

Let $\rho$ be the natural representation of $H$ on $V_{i_1}$. then $\Theta = \mathrm{Ind}_H^G \rho$, and (b) holds. Notice that $\rho$ must be irreducible (if this were not the case, then $\Theta = \mathrm{Ind}_H^G \rho$ would also be reducible): $\square$

Remark - Notice that the possibilities (a) and (b) cannot co-exist. Suppose that $\mathrm{Res}_A \Theta = \Gamma^{\oplus e}$, and that $\exists A \leq H \lneq G$ and and irreducible representation of $G$ s.t. $\Theta = \mathrm{Ind}_H^G \rho$.
Then $\mathrm{Res}_H \Theta$ contains $\rho$, and $\mathrm{Res}_A \rho$ is a direct sum of copies of $\Gamma$: $\mathrm{Res}_A \rho = \Gamma^{\oplus a}$.

Case 1: $a > 1$. then $\rho$ contains $a(>1)$-many copies of $\mathrm{Ind}_A^H \Gamma$. We reach a contradiction because $\rho$ is irreducible.

Case 2: $a = 1$. Then $\mathrm{Res}_A \rho = \Gamma$. We notice that $\rho$ contains $\mathrm{Ind}_A^H \Gamma$, but $\rho$ has dimension equal to $\dim(\Gamma)$. We reach a contradiction unless $H = A$ and $\rho = \Gamma$. If $\Theta = \mathrm{Ind}_A^G \Gamma$, then $\mathrm{Res}_A \Theta$ can only contain one copy of $\Gamma$ and therefore must equal $\Gamma$; but we can't have $\begin{cases} \Theta = \mathrm{Ind}_A^G \Gamma \\ \mathrm{Res}_A \Theta = \Gamma \end{cases}$ for dimensional reasons. $\square$

Example Choose $G = D_n$ ; $A = C_n \trianglelefteq D_n$. Choose a primitiv[e] $n$th root of unity $\omega$, and write $D_n = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Assume $n$ odd

| Representations of $D_n$ | Restriction to $C_n$ | |
|---|---|---|
| $\mu_0 : D_n \to \mathbb{C}^*$, $a^i b^j \mapsto 1$ | $\chi_0 = \text{triv.}$ | $\to$ isotypic |
| $\mu_1 : D_n \to \mathbb{C}^*$, $a^i b^j \mapsto (-1)^j$ | $\chi_0 = \text{triv.}$ | $\to$ isotypic |
| $\theta_k : D_n \to GL(\mathbb{C}^2)$, $k = 1 \ldots \frac{n-1}{2}$; $a \mapsto \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{bmatrix}$, $b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\chi_k \oplus \chi_{-k}$; we choose $\chi_k : C_n \to \mathbb{C}^*$, $a \mapsto \omega^k$ | $\to$ not isotypic. Choose $H = A = C_n$. Then $\theta_k = \mathrm{Ind}_{C_n}^{D_n} \chi_k \left( = \mathrm{Ind}_{C_n}^{D_n} \chi_{-k} \right)$. |

You can't write $\mu_1$ or $\mu_0$ as $\mathrm{Ind}_H^G \rho$, with $A = C_n \leq H \lneq G$.
[$\dim \mu_1 = \dim \mu_0 = 1$ so you would need $\frac{|G|}{|H|} = 1$ ...

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2. Restriction to subgroups

Let $H, K$ be subgroups of $G$ ($\leftarrow$ any two subgroups!).
Let $(\rho, W)$ be an irreducible representation of $H$.
In this section we describe $\mathrm{Res}_K \mathrm{Ind}_H^G W$.

For each $s$ in $G$, write

$$KsH = \{ksh : k \in K, h \in H\}$$

for the double $(H\text{-}K)$-coset of $s$.

Choose a set of representives $S$ for the double $(H\text{-}K)$-cosets, so that

$$G = \bigoplus_{s\in S} KsH. \quad \left[\text{We also write } S = K\backslash G/H\right].$$

- For each $s\in S$, define $H_s = (sHs^{-1})\cap K \leq K$, and consider the representation $(\rho^s, W^s)$ of $H_s$ defined by:
  - $W^s = W$
  - $\rho^s(x) = \rho(s^{-1}xs) \quad \forall x\in H_s$

[because $x = shs^{-1}$ for some $h\in H$, and $\rho$ is a repr. of $H$, $\rho(s^{-1}xs)$ is well defined. Moreover:

$$\rho^s(x_1x_2) = \rho(s^{-1}x_1x_2s) = \rho(\underbrace{s^{-1}x_1s}_{\text{in } H}\ \underbrace{s^{-1}x_2s}_{\text{in } H}) =$$

$$= \rho(s^{-1}x_1s)\rho(s^{-1}x_2s) = \rho^s(x_1)\rho^s(x_2) \quad \forall x_1, x_2\in H_s,$$

so $(\rho^s, W^s)$ is a well defined representation of $H_s$].

- Notice that $H_s$ is a subgroup of $K$, hence we can ~~the~~ induce the representation $(\rho^s, W^s)$ from $H_s$ to $K$.

The claim is that $\mathrm{Res}_K(\mathrm{Ind}_H^G W)$ is the direct sum of the various $\mathrm{Ind}_{H_s}^K(W^s)$, with $s\in S = K\backslash G/H$.

<u>Proposition</u>: $\mathrm{Res}_K(\mathrm{Ind}_H^G W) = \bigoplus_{s\in K\backslash G/H} \mathrm{Ind}_{H_s}^K(W^s)$.

<u>proof</u> - We look at the vector space $V$ of $\mathrm{Ind}_H^G W$, and we decompose it as a direct sum of $K$-invariant subspaces. Then we show that those $K$-invariant subspaces are of the form $\mathrm{Ind}_{H_s}^K(W^s)$, with $s\in S = K\backslash G/H$.

We can write:

$$V = \mathrm{Ind}_H^G W = \bigoplus_{x \in G/H} xW =$$

$$= \bigoplus_{s \in K\backslash G/H} \left( \bigoplus_{x \in G/H \,:\, x \in KsH} xW \right)$$

$$= \bigoplus_{s \in K\backslash G/H} V(s).$$

We have set $V(s) = \bigoplus_{x \in G/H \,:\, x \in KsH} xW.$

Notice that

- $V(s)$ is stable under $K$. So $V = \bigoplus_{s \in K\backslash G/H} V(s)$ is a decomposition of $\mathrm{Res}_K V$ as a direct sum of $K$-stable subspaces.
- If $x \in G/H$ and $x \in KsH$, then $x = k_1 s h_1$ for some $h \in H$ and $k$ in $K$. Notice that

$$k_1 s h_1 = k_2 s h_2 \iff k_2^{-1} k_1 s = s h_2 h_1^{-1} \iff$$

$$\iff k_2^{-1} k_1 = s(h_2 h_1^{-1}) s^{-1} \in [sHs^{-1} \cap K] = H_s.$$

So $k_1 s H = k_2 s H \iff k_1 H_s = k_2 H_s$.

We can therefore write:

$$V(s) = \bigoplus_{y \in K/H_s} ysW.$$

Finally, we notice that sW is a representation $sW$ of $H_s$:

if $h^s = shs^{-1}$, with $h \in H$, then

$$h^s \cdot (s\underline{w}) = shs^{-1} \cdot s\underline{w} = s \cdot h\underline{w}.$$

the representation $sW$ is isomorphic to $W^s$, and the map $T: W^s \longrightarrow sW$, $\underline{w} \longrightarrow s\underline{w}$ is an intertwining operator.

$$\begin{tikzcd}
W^s \arrow[r, "T"] \arrow[d, "h^s"'] & sW \arrow[d, "h^s"] \\
W^s \arrow[r, "T"] & sW
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{w} \arrow[r] \arrow[d] & s\underline{w} \arrow[d] \\
s^{-1} h^s s w \arrow[r] & s(s^{-1}h^s s w) = h^s s w
\end{tikzcd}$$

commutative diagram $\forall h^s \in H_s$

It follows that $V(s) \cong \bigoplus_{y \in K/H_s} yW^s = \mathrm{Ind}_{H_s}^{K} W^s$.

We get:

$$V = \bigoplus_{s \in {}_K\backslash G/H} V(s) = \bigoplus_{s \in \backslash G/K} \mathrm{Ind}_{H_s}^{K} W^s. \qquad \blacksquare$$

as K-representation

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SPECIAL CASE: Assume that $H \trianglelefteq G$ and that $H = K$.

Then $H_s = (sHs^{-1}) \cap K = \underbrace{sHs^{-1}}_{=H} \cap H = H$, and

$\rho^s$ is just a "conjugate" of $\rho$: $\rho^s(h) = \rho(s^{-1}hs)$ $\forall h \in H$

· the proposition gives:

$$\operatorname{Res}_H \left( \operatorname{Ind}_H^G \rho \right) = \bigoplus_{s \in {}_H\backslash G/H} \operatorname{Ind}_{H_s = H}^{K=H} \rho^s \underset{H \text{ normal}}{=} \bigoplus_{x \in G/H} \operatorname{Ind}_H^H \rho^x = \bigoplus_{x \in G/H} \rho^x .$$

When $H \trianglelefteq G$, then

$$\operatorname{Res}_H \left( \operatorname{Ind}_H^G \rho \right) = \bigoplus_{x \in G/H} \rho^x .$$

direct sum of conjugates of $\rho$ !!!

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## 3. Mackey's irreducibility criterion

We apply the results from the previous section to the case in which $H = K$, but $H$ is not necessarily normal, and derive a criterion for the irreducibility of $\operatorname{Ind}_H^G \rho$.

Fix an irreducible representation $(\rho, W)$ of $H$.

For all $s$ in $G$, define $H_s = (sHs^{-1}) \cap H \leq H$

and $(\rho^s, W^s)$ by:

- $W^s = W$
- $\rho^s(h^s) = \rho(s^{-1} h^s s) \quad \forall h^s \in H_s$.

Then $(\rho^s, W^s)$ is a representation of $H_s$.

Because $H_s \le H$, we can also look at the restriction of $\rho$ to $H_s$, that we denote by $\mathrm{Res}_{H_s} \rho = \rho_{H_s}$

the two representations $\rho^s$ and $\rho_{H_s}$ of $H_s$ may or may not have common factors, and this property is extremely related to the irreducibility of $\mathrm{Ind}_H^G \rho$.

Indeed, "Mackey's irreducibility criterion" holds:

<u>Proposition</u> In order for the induced representation $\mathrm{Ind}_H^G \rho$ be irreducible, it is necessary and sufficient that the following two conditions be satisfied:

① $\rho$ is irreducible

② For each $s \notin H$, the two representations $\rho^s$ and $\rho_{H_s}$ of $H_s$ are disjoint [i.e. they have no common factors].

<u>proof</u>- For brevity of notations, set $m(\Theta_1, \Theta_2) = \langle \chi_{\Theta_1}, \chi_{\Theta_2} \rangle$ for every pair of representations...

then we can write:

$$\mathrm{Ind}_H^G \rho \text{ irreducible} \iff m(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_H^G \rho) = 1$$

$$\underset{\text{Frob. reciprocity}}{\iff} \quad m(\rho, \mathrm{Res}_H \mathrm{Ind}_H^G \rho) = 1$$

$$\iff \quad m\left(\rho, \bigoplus_{s \in H \backslash G / H} \left(\mathrm{Ind}_{H_s}^H \rho^s\right)\right) = 1$$