LECTURE 12

In this lecture we give a formula for the character of induced representation. Many properties of induced representation follow easily from this formula....

1. FROBENIUS CHARACTER FORMULA

Let (τ, W) be a representation of $H < G$, and let (ρ, V) be the corresponding induced representation of G: $\rho = \mathrm{Ind}_H^G \tau$.

For $s \in G$, we compute $\chi_\rho(s)$.

Write C for the conjugacy class of s in G, and

$$C \cap H = D_1 \uplus D_2 \uplus .. \uplus D_\ell$$

for the decomposition of $C \cap H$ into H-conjugacy classes.

We will show that

$$\chi_\rho(s) = \chi_\rho(C) = \frac{|G|}{|H|} \sum_{j=1}^{\ell} \frac{\# C}{\# D_j} \chi_\tau(D_j).$$

In particular, $\chi_\rho(C) = 0$ if $C \cap H = \phi$.

Write $V = \bigoplus_{i=1}^{k} g_i W$, with $g_1, .., g_k$ representatives in G for the cosets of H (i.e. $G = g_1 H \uplus g_2 H \uplus \dots \uplus g_k H$).

Then $\rho(s)$ permutes the subspaces $g_i W$'s. More precisely:

$$\rho(s) : g_i W \longrightarrow g_{i'} W$$

if $s g_i H = g_{i'} H$ $(\Longleftrightarrow g_{i'}^{-1} s g_i \in H)$.

It follows that $g_i W$ contributes to the trace if and only if $g_i^{-1} s g_i \in H$. Suppose that this is the case, and write $g_i^{-1} s g_i = h$, i.e. $s g_i = g_i h$. then

$$\rho(s) : g_i W \to g_i W, \quad g_i \underline{w} \mapsto g_i \tau(h) \underline{w}.$$

f $s g_i = g_i h$, Then $\rho(s)$ acts on $g_i W$ exactly in The same way n which $\tau(h) = \tau(g_i^{-1} s g_i)$ acts on W.

It follows That:

$$\boxed{\chi_\rho(s) = \sum_{\substack{i=1\dots k \\ g_i^{-1} s g_i \in H}} \chi_\tau(g_i^{-1} s g_i)}$$

this first formulation of Frobenius character formula is not very suitable for practical use. We are going To rewrite iT in a much more convenient fashion ....

the first change we want To make is to sum over all $\underline{g \in G}$ s.t. $g^{-1} s g \in H$, instead of summing over the $\underline{g \in \{g_1, \dots, g_k\}}$ with This property.

We notice That if $g \in G$, Then $g = g_i h$ for some $h \in H$ and clearly $g^{-1} s g \in H$ if and only if $g_i^{-1} s g_i \in H$.

For all $h \in H$, $\chi_\tau(g_i^{-1} s g_i) = \chi_\tau((g_i h)^{-1} s (g_i h))$. So

$$\sum_{\substack{g \in G \\ g^{-1} s g \in H}} \chi_\tau(g^{-1} s g) = \sum_{\substack{i=1\dots k \\ g_i^{-1} s g_i \in H}} \sum_{h \in H} \chi_\tau((g_i h)^{-1} s (g_i h)) =$$

$$= \sum_{\substack{i=1\dots k \\ g_i^{-1} s g_i \in H}} |H| \, \chi_\tau(g_i^{-1} s g_i).$$

$$\Rightarrow \chi_\rho(s) = \frac{1}{|H|} \sum_{g \in G,\, g^{-1} s g \in H} \chi_\tau(g^{-1} s g).$$

As g runs over $\{g \in G : g^{-1} s g \in H\}$, then $g^{-1} s g$ runs over $C \cap H$, where C is the conjugacy class of s in G.

Write $C \cap H = \biguplus_{j=1}^{e} D_j$ for the decomposition of $C \cap H$ into a disjoint union of H-conjugacy classes.

For each $j = 1 \ldots e$, and each x in D_j, there are exactly $\frac{|G|}{\# C}$-many g's in G st $g^{-1} s g = x$. [Because $\frac{|G|}{\# C}$ is the order of the stabilizer of s in G].

So we can write:

$$\chi_{\rho}(s) = \frac{1}{|H|} \sum_{g \in G : g^{-1} s g \in H} \chi_{r}(g^{-1} s g) = \frac{1}{|H|} \sum_{\substack{g \in G : \\ g^{-1} s g \in H \cap C = \biguplus_{j=1}^{e} D_j}} \chi_{r}(g^{-1} s g) =$$

$$= \frac{1}{|H|} \sum_{j=1}^{e} \sum_{x \in D_j} \sum_{g \in G : g^{-1} s g = x} \chi_{r}(x) =$$

$$= \frac{1}{|H|} \sum_{j=1}^{e} \sum_{x \in D_j} \frac{|G|}{\# C} \underbrace{\chi_{r}(x)}_{= \chi_{r}(D_j)} = \frac{|G|}{|H|} \sum_{j=1}^{e} \frac{\# D_j}{\# C} \chi_{r}(D_j).$$

This new formulation of the Frobenius character formula is the one that is most useful for computations...

FROBENIUS CHARACTER FORMULA

$$\chi_{\mathrm{Ind}_H^G r}(C) = \begin{cases} 0 & \text{if } C \cap H = \emptyset \\ \frac{|G|}{|H|} \sum_{j=1}^{e} \frac{\# D_j}{\# C} \chi_{r}(D_j) & \text{if } C \cap H = \biguplus_{j=1}^{e} D_j \end{cases}$$ (← H-conjugacy classes)

Example – Let $G = S_4$ and let $H \leq G$ be the subgroup of S_4 generated by (123).

Compute the character of the induced representations from H to G.

· We notice that $H \cong C_3$ and it has 3 one-dimensional characters:

	1	(123)	(132)
χ_{Γ_1}	1	1	1
χ_{Γ_2}	1	ω	$\bar{\omega}$
χ_{Γ_3}	1	$\bar{\omega}$	ω

[We have set $\omega = e^{2\pi i/3}$].

The group $G = S_4$ has 5 conjugacy classes:

$C_1 = \{1\}$

$C_2 = \{\text{transpositions}\} = \{(--)\}$

$C_3 = \{\text{3-cycles}\} = \{(---)\}$

$C_4 = \{\text{4-cycles}\} = \{(----)\}$

$C_5 = \{\text{product of 2 disjoint cycles}\} = \{(--)(--)\}$.

the group $H \cong C_3$ has 3 conjugacy classes:

$D_1 = \{1\}$

$D_2 = \{(123)\}$

$D_3 = \{(132)\}$.

Clearly, have: ——

$$\left[\begin{array}{l} C_1 \cap H = D_1 \\ C_2 \cap H = \emptyset \\ C_3 \cap H = D_2 \uplus D_3 \\ C_4 \cap H = C_5 \cap H = \emptyset. \end{array}\right.$$

Therefore, for all $i=1..3$, we have:

- $\chi_{\mathrm{Ind}_H^G r_i}(C) = 0$ if $C = C_2, C_4$ or C_5

- $\chi_{\mathrm{Ind}_H^G r_i}(C_1) = \frac{|G|}{|H|}\frac{\#D_1}{\#C_1}\underbrace{\chi_{r_i}(D_1)}_{=1} = \frac{|G|}{|H|}\frac{\#D_1}{\#C_1} = \frac{24}{3}\cdot\frac{1}{1} = 8.$

- $\chi_{\mathrm{Ind}_H^G r_i}(C_3) = \frac{|G|}{|H|}\left[\frac{\overset{=1}{\#D_2}}{\#C_3}\chi_{r_i}(D_2) + \frac{\overset{=1}{\#D_3}}{\#C_3}\chi_{r_i}(D_3)\right] =$

$$= \frac{24}{3}\,\frac{1}{\#C_3}\left[\chi_{r_i}(D_2) + \chi_{r_i}(D_3)\right]$$

$$= \frac{\cancel{24}}{\cancel{3}}\cdot\frac{1}{\cancel{8}}\left[\chi_{r_i}(D_2) + \chi_{r_i}(D_3)\right] = \left[\chi_{r_i}(D_2) + \chi_{r_i}(D_3)\right].$$

The size of the stabilizer of in S_4 is $3\cdot 1! = 3$. So the size of the conjugacy class is $\frac{24}{3} = 8$.

Notice that $C_3 \cap H = \{(123), (132)\}$ and has cardinality 2 $(= \#D_2 + \#D_3)$.

The character of the induced representations are therefore:

	C_1	C_2	C_3	C_4	C_5
$\mathrm{Ind}\, r_1$	8	0	2	0	0
$\mathrm{Ind}\, r_2$	8	0	-1	0	0
$\mathrm{Ind}\, r_3$	8	0	-1	0	0

COROLLARIES

① For all representations r_1 and r_2 of $H \leq G$,
$$\mathrm{Ind}_H^G (r_1 \oplus r_2) = \mathrm{Ind}_H^G r_1 \oplus \mathrm{Ind}_H^G r_2 .$$

② If θ is any representation of G and r is any representation of H, then
$$\theta \otimes (\mathrm{Ind}_H^G r) = \mathrm{Ind}_H^G (\mathrm{Res}_H^G \theta \otimes r).$$

proof - ① $\chi_{\mathrm{Ind}_H^G (r_1 \oplus r_2)}(C) = \frac{|G|}{|H|} \sum_{j=1}^{e} \frac{\# D_j}{\# C} \chi_{r_1 \oplus r_2}(D_j) =$

$C \cap H = \biguplus_{j=1}^{e} D_j$

$$= \frac{|G|}{|H|} \sum_{j=1}^{e} \frac{\# D_j}{\# C} \left(\chi_{r_1}(D_j) + \chi_{r_2}(D_j)\right) =$$

$$= \left[\frac{|G|}{|H|} \sum_{j=1}^{e} \frac{\# D_j}{\# C} \chi_{r_1}(D_j)\right] + \left[\frac{|G|}{|H|} \sum_{j=1}^{e} \frac{\# D_j}{\# C} \chi_{r_2}(D_j)\right] =$$

$$= \chi_{\mathrm{Ind}_H^G r_1}(C) + \chi_{\mathrm{Ind}_H^G r_2}(C).$$

② $\chi_{\theta \otimes \mathrm{Ind}_H^G r}(C) = \chi_\theta(C)\, \chi_{\mathrm{Ind}_H^G r}(C) =$

$$= \chi_\theta(C) \left[\sum_{j=1}^{e} \frac{|G|}{|H|} \frac{\# D_j}{\# C} \chi_r(D_j)\right] =$$

$$= \frac{|G|}{|H|} \sum_{j=1}^{e} \frac{\# D_j}{\# C} \chi_r(D_j) \overbrace{\overline{\chi_\theta(C)}}^{\overline{\chi_{\theta_{|H}}(D_j)}\ \forall j=1\ldots e \text{ b.c. } C \supseteq \biguplus_{j=1}^{e} D_j} =$$

6.

$$= \frac{|G|}{|H|} \sum_{j=1}^{e} \frac{\# D_j}{\# C} \chi_{\theta_H \otimes r}(D_j) =$$

$$= \chi_{\mathrm{Ind}_H^G(\theta_H \otimes r)}(C). \checkmark$$

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2. FROBENIUS RECIPROCITY

Let $H \leq G$ be a subgroup of G. Let (r, W) be any representation of H and let (θ, U) be any representation of G. Then

$$\langle \chi_{\mathrm{Ind}_H^G r}, \chi_\theta \rangle_G = \langle r, \chi_{\mathrm{Res}_H^G \theta} \rangle_H .$$

<u>proof</u> - $\langle \chi_{\mathrm{Ind}_H^G r}, \chi_\theta \rangle_G = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\theta(g)} \chi_{\mathrm{Ind}_H^G r}(g) =$

$$= \frac{1}{|G|} \sum_{C \text{ (conjugacy classes of } G)} \# C\, \overline{\chi_\theta(C)}\, \chi_{\mathrm{Ind}_H^G r}(C) =$$

$$= \frac{1}{\cancel{|G|}} \frac{\cancel{|G|}}{|H|} \sum_{C} \sum_{\substack{D_j \subseteq C \\ H \text{ conj. classes}}} \cancel{\# C} \frac{\# D_j}{\cancel{\# C}} \underbrace{\overline{\chi_\theta(C)}}_{\overline{\chi_{\theta_H}(D_j)}} \chi_r(D_j) =$$

$$= \frac{1}{|H|} \sum_{C} \sum_{D_j \subseteq C} \# D_j \overline{\chi_{\theta_H}(D_j)} \chi_r(D_j) = \frac{1}{|H|} \sum_{C} \sum_{x \in D_j \subseteq C \cap H} \overline{\chi_{\theta_H}(D_j)} \chi_r(D_j)$$

$$= \frac{1}{|H|} \sum_{h \in H} \overline{\chi_{\theta_H}(D_j)}\, \chi_{\nu}(D_j) = \langle \chi_{\theta_H}, \chi_{\nu} \rangle_H =$$

$\bigcup_C C \cap H = H$

$$= \langle \chi_\nu, \chi_{\mathrm{Res}^G_H(\theta)} \rangle_H . \checkmark$$

FROBENIUS RECIPROCITY

$$m(\theta, \mathrm{Ind}^G_H \nu) = m(\nu, \mathrm{Res}^G_H \theta)$$

For every irred. repr. θ of G and ν of H, the multiplicity of θ in $\mathrm{Ind}^G_H \nu$ is equal to the multiplicity of ν in $\mathrm{Res}^G_H \nu$.

As a corollary, we obtain the principle of <u>induction in stages</u>: If $H \le K \le G$ and ν is a repr. of H, then $\mathrm{Ind}^G_H \nu = \mathrm{Ind}^G_K (\mathrm{Ind}^K_H \nu)$.

<u>proof</u> - It's enough to show that for every irreducible repr. θ of G, $m(\theta, \mathrm{Ind}^G_H \nu) = m(\theta, \mathrm{Ind}^G_K \mathrm{Ind}^K_H \nu)$. this result will follow from Frobenius reciprocity. Indeed:

$$m(\theta, \mathrm{Ind}_K^G \mathrm{Ind}_H^K \rho) =$$

$$= \langle \chi_\theta , \chi_{\mathrm{Ind}_K^G(\mathrm{Ind}_H^K \rho)} \rangle_G = \langle \chi_{\mathrm{Res}_K^G \theta} , \chi_{\mathrm{Ind}_H^K \rho} \rangle_K =$$

$$= \langle \chi_{\underbrace{\mathrm{Res}_H^K \mathrm{Res}_K^G \theta}_{\overset{\shortparallel}{\mathrm{Res}_H^G \theta}}} , \chi_\rho \rangle_H =$$

$$= \langle \chi_{\mathrm{Res}_H^G \theta} , \chi_\rho \rangle_H = m(\rho, \mathrm{Res}_H^G \theta) = m(\theta, \mathrm{Ind}_H^G \rho).$$

✓

Next, we show how to apply Frobenius reciprocity to compute character of induced representations ....

- $G = S_4$
- $H \leq S_4$, The subgroup generated by (1234).
- ρ_i = The (one-dimensional) representations of H $(i = 1..4)$.

Recall That:

character table of S_4 =>

	1	(12)	(12)(34) (in H)	(1234) (in H)	(123)
ρ_1	1	1	1	1	1
ρ_2	1	−1	1	−1	1
ρ_3	2	0	2	0	−1
ρ_4	3	−1	−1	1	0
ρ_5	3	1	−1	−1	0

character table of $H \cong C_4$

	1	(13)(24)	(1234)	(1432)
Γ_1	1	1	1	1
Γ_2	1	-1	i	$-i$
Γ_3	1	1	-1	-1
Γ_4	1	-1	$-i$	i

Restrictions :

	1	(13)(24)	(1234)	(1432)
$\mathrm{Res}\,\rho_1$	1	1	1	1
$\mathrm{Res}\,\rho_2$	1	1	-1	-1
$\mathrm{Res}\,\rho_3$	2	2	0	0
$\mathrm{Res}\,\rho_4$	3	-1	1	1
$\mathrm{Res}\,\rho_5$	3	-1	-1	-1

So we get:

$$\rho_1 \xrightarrow{\text{RESTRICTION}} \Gamma_1$$
$$\rho_2 \longrightarrow \Gamma_3$$
$$\rho_3 \longrightarrow \Gamma_1 + \Gamma_3$$
$$\rho_4 \longrightarrow \Gamma_1 + \Gamma_2 + \Gamma_4$$
$$\rho_5 \longrightarrow \Gamma_3 + \Gamma_2 + \Gamma_4 .$$

By Frobenius reciprocity, we find :

$$\Gamma_1 \xrightarrow{\text{INDUCTION}} \rho_1 \oplus \rho_3 \oplus \rho_4$$
$$\Gamma_2 \longrightarrow \rho_4 \oplus \rho_5$$
$$\Gamma_3 \longrightarrow \rho_2 \oplus \rho_3 \oplus \rho_5$$
$$\Gamma_4 \longrightarrow \rho_4 \oplus \rho_5 .$$

3. Induction from a normal subgroup of index 2.

Let $H \leq G$ be a normal subgroup of index 2. Let $\lambda : G \to \mathbb{C}^*$, $g \mapsto \begin{cases} 1 & \text{if } g \in H \\ -1 & \text{if } g \notin H \end{cases}$

Let V be an irreducible representation of G and let $W = \mathrm{Res}_H^G V$. Then exactly one of the following holds:

① $\boxed{\chi_V \neq \chi_V \cdot \lambda} \iff \boxed{\chi_V(g) \neq 0 \text{ for some } g \in G - H} \iff \boxed{W = \mathrm{Res}_H^G V \text{ is irreducible}}$,

and $\mathrm{Ind}_H^G W = V \oplus V'$ (where $\chi_{V'} = \chi_V \cdot \lambda$).

② $\boxed{\chi_V = \chi_V \cdot \lambda} \iff \boxed{\chi_V|_{G-H} \equiv 0} \iff \boxed{W = \mathrm{Res}_H^G V = W_1 \oplus W_2}$

(not isom. reprs of H of the same dim.)

and $\mathrm{Ind}_H^G W_1 = \mathrm{Ind}_H^G W_2 = V$.

Example

Recall the character tables of S_4 and A_4:

S_4	1	(12)	$(12)(34)$ ↓ in H	(123) ↓ in H	(1234)
ρ_1	1	1	1	1	1
ρ_2	1	-1	1	1	-1
ρ_3	2	0	2	-1	0
ρ_4	3	-1	-1	0	1
ρ_5	3	1	-1	0	-1

	1	(12)(34)	(123)	(132)
λ_1	1	1	1	1
λ_2	3	-1	0	0
λ_3	1	1	ω	$\bar{\omega}$
λ_4	1	1	$\bar{\omega}$	ω

- $\chi_{\rho_1} \cdot \lambda = \chi_{\rho_2} \neq \chi_{\rho_1}$

  $\mathrm{Res}_{A_4}(\rho_1) = \mathrm{Res}_{A_4}(\rho_2) = \lambda_1$

  $\mathrm{Ind}_{A_4}^{S_4} \lambda_1 = \rho_1 \oplus \rho_2$

- $\chi_{\rho_3} = \chi_{\rho_3} \cdot \lambda$

  $\mathrm{Res}_{A_4} \rho_3 = \lambda_3 + \lambda_4$

  $\mathrm{Ind}_{A_4}^{S_4} \lambda_3 = \mathrm{Ind}_{A_4}^{S_4} \lambda_4 = \rho_3$

- $\chi_{\rho_4} \cdot \lambda = \chi_{\rho_5}$

  $\mathrm{Res}_{A_4} \rho_4 = \mathrm{Res}_{A_4} \rho_5 = \lambda_2$

  $\mathrm{Ind}_{A_4}^{S_4} \lambda_2 = \rho_4 \oplus \rho_5$.

Suggested Problems

(1) For every irreducible representation ρ of S_4, find the character of $\mathrm{Ind}_{S_4}^{S_5} \rho$. [Hint: use Frobenius character formula].

(2) Let $G = S_4$, let $H \simeq D_8$ $\left(H = \langle (12), (1234) \rangle \right)$.
Find $\mathrm{Res}_{D_8}^{S_4} \theta$, for every irreducible representation θ of S_4; Then find $\mathrm{Ind}_{D_8}^{S_4} \tau$ for every irreducible representation τ of D_8. [Hint: use Frobenius reciprocity].

(3) a. Find the character table of S_5

b. Use restrictions to find the character table of A_5.

c. Use Frobenius reciprocity to find $\mathrm{Ind}_{A_5}^{S_5} \tau$, for every irreducible representation τ of A_5.