Review from last lecture:

Assume that $H \trianglelefteq G$ of index 2. Let ρ be an irreducible representation of G, with character χ_ρ.

Let λ be the character of G induced by the non trivial character of G/H: $\lambda(g) = \begin{cases} 1 & \text{if } g \in H \\ -1 & \text{if } g \in G-H. \end{cases}$

then two possibilities can occur:

① $\rho_H = \mathrm{Res}^G_H \rho$ is irreducible. Then $\chi_\rho \neq \chi_\rho \cdot \lambda$ and χ_ρ and $\chi_\rho \cdot \lambda$ are the unique irreducible characters of G that restrict to χ_{ρ_H}.

② $\rho_H = \mathrm{Res}^G_H \rho$ is reducible. Then $\rho_H = \varphi_1 \oplus \varphi_2$, with φ_1 and φ_2 inequivalent representations of H of the same dimension. Moreover, $\chi_\rho = \chi_\rho \cdot \lambda$ and there is no other irreducible representation of G whose restriction to H contains φ_1 and/or φ_2.

Application: Knowing that A_7 has 9 conjugacy classes and that the irred. reprs of S_7 have dim. 1, 1, 6, 6, 14, 14, 14, 14, 15, 15, 20, 21, 21, 35, 35; find the complete list of degrees of the irreducible char. of A_7.

▸ If $\deg \chi_\rho$ is odd $\Rightarrow \chi_{\rho_H}$ must be irreducible $\Rightarrow \chi_\rho \neq \chi_\rho \cdot \lambda$.

Induced Representations

In This section we study induced representations from a subgroup. We have seen in The previous lecture That every representation of G restricts To a ~~representation~~ representation of $H \leq G$ on The same space. We are going To reverse This construction, and produce a representation of G for each given representation of $H \leq G$.

There are 2 ways To define induce representations. We discuss all of Them, and we show Their equivalence.

DEFINITION #1 Let H be a subgroup of G (not necessarily normal) and let (τ, W) be a representation of H.
Consider The vector space

$$X = \{ f : G \to W,\ f(gh) = \tau(h^{-1}) f(g)\ \forall g \in G, h \in H \}.$$

[It consists of functions from G To W ($\leftarrow$ The representation space of τ) that behave "well" under right Translation by H.]
We define a representation ρ of G on X by:

$$[\rho(s) f](g) = f(s^{-1} g),\quad \forall f \in X, \forall g, s \in G.$$

Elements of G act on X by left Translation. To check that this action is well defined, we prove That

$$[\rho(s) f](gh) = \tau(h^{-1}) [\rho(s) f(g)] \quad \forall s, g \in G, \forall h \in H, \forall f \in X.$$

Then $f \in X$ and $\varphi(f) = (\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k)$.
So φ is a vector space isomorphism, and therefore

$$\dim(X) = \dim(W^{\oplus k}) = k \dim W = \frac{|G|}{|V|} \dim \tau.$$

Let's construct an explicit basis of X.
Fix a basis $\underline{w}_1, \dots, \underline{w}_n$ of W (of course $n = \dim \tau$). For all $j = 1, \dots, k = \frac{|G|}{|H|}$ and all $l = 1, \dots, n = \dim W$, define an element of X by:

$$f_{jl} : \bigoplus_{i=1}^{k} g_i H \longrightarrow W, \quad g_i h \longrightarrow \delta_{ij} \tau(h^{-1}) \underline{w}_l .$$

This function is identically zero on the cosets $g_i H$ with $i \neq j$, and takes the values

$$f_{jl}(g_j h) = \tau(h^{-1}) \underline{w}_l$$

on $g_j H$. In other words, f_{jl} is the inverse image via φ of the k-upla $(\underline{0}, \underline{0}, \dots, \underset{\substack{\uparrow \\ j}}{\underline{w}_l}, \underline{0}, \dots, \underline{0}) \in W^{\oplus k}$.

It's clear that this is a basis, because $\{\varphi(f_{jl})\}_{\substack{j=1\dots k \\ l=1\dots n}}$ are a basis of $W^{\oplus k}$ and φ is an isomorphism.

DEFINITION #2 Let (τ, W) be a representation of H.
We define a representation (θ, V) of G which is induced from (τ, W).
As a vector space, $V = \bigoplus_{i=1}^{k} g_i W$.

$\rightarrow \left[\rho(s) f\right](gh) = f(s^{-1}gh) = \tau(h^{-1})\left[f(s^{-1}g)\right] = \tau(h^{-1})\left[\rho(s) f(g)\right]$ ✓

- We call (ρ, X) the induced representation of (τ, W) from H to G, and we write $\rho = \mathrm{Ind}_H^G \tau$.

<u>Remark</u>: $\dim\left(\mathrm{Ind}_H^G \tau\right) = \dim\tau \cdot [G:H] = \dim\tau \, \dfrac{|G|}{|H|}$.

<u>Proof</u>- Choose representatives $g_1, \dots, g_k \in G$ for the left cosets, so that

$$G = g_1 H \uplus g_2 H \uplus \dots \uplus g_k H,$$

[We have denoted by $\uplus$ the disjoint union of sets].

By construction, $k = [G:H] = \dfrac{|G|}{|H|}$.

- We notice that every element $f \in X$ is completely determined by its values at $g_1, g_2, \dots, g_k$.

Indeed, for all $s \in G$, we can write

$$f(s) \underset{\uparrow}{=} f(g_i h) = \tau(h^{-1}) f(g_i).$$

$s = g_i h$ for some $h \in H$ and $i = 1 \dots k$

It follows that the map

$$\varphi : X \longrightarrow W^{\oplus k}, \quad f \mapsto (f(g_1), f(g_2), \dots, f(g_k))$$

is injective. It's easy to show that φ is also surjective:

- if $\underline{v}_1, \dots, \underline{v}_k$ are any elements of W, we can define

$$f : G = \biguplus_{i=1}^{k} g_i H \longrightarrow W, \quad g_i h \mapsto \tau(h^{-1}) \underline{v}_i.$$

Remember that $g_1, \dots, g_k$ are representatives for the left cosets of H in G (so that $G = \biguplus_{i=1}^{k} g_i H$).

For all $i = 1..k$, $g_i W$ is a "formal" vector space with operations $g_i \underline{w}_1 + g_i \underline{w}_2 = g_i(\underline{w}_1 + \underline{w}_2)$ and $c(g_i \underline{w}) = g_i c\underline{w}$. Clearly, $g_i W$ is isomorphic to W as a vector space.

If H is normal, $g_i W$ carries a representation of H defined by $h \cdot g_i \underline{w} = g_i\, r(g_i^{-1} h g_i)\underline{w}$, but - in general - we won't assume H to be normal.

So $g_i W$ should just be regarded a vector space.

We let G act on $V = \bigoplus_{i=1}^{k} g_i W$ as follows:

if $s \in G$, there is a unique index $i' \in \{1..k\}$ st $s g_i \in g_{i'} H$ (say that $s g_i = g_{i'} h$), then we set

$$\theta(s)(g_i \underline{w}) \underset{\substack{\uparrow \\ s g_i = g_{i'} h}}{=} g_{i'}\, t(h)\, \underline{w}.$$

Notice that $\theta(s)$ permutes the subspaces $\{g_i W\}_{i=1..k}$ of V, and carries $g_i W$ into $g_{i'} W$ if $s g_i \in g_{i'} H$.

Let's verify that θ is a well defined representation of G on V: $\theta(s_1)\theta(s_2) g_i \underline{w} = \theta(s_1 s_2) g_i \underline{w}$ $\forall s_1, s_2 \in G$ $\forall \underline{w} \in W$, $\forall i = 1..k$.

If $s_2 g_i = g_{i_2} h_2$, and $s_1 g_{i_2} = g_{i_1} h_1$, Then

$$(s_1 s_2) g_i = s_1(g_{i_2} h_2) = (s_1 g_{i_2}) h_2 = (g_{i_1} h_1) h_2 = g_{i_1}(h_1 h_2)$$

So we can write:

$$\Theta(s_2) g_i \underline{w} \underset{\substack{\uparrow \\ s_2 g_i = g_{i_2} h_2}}{=} g_{i_2} \underbrace{(r(h_2)\underline{w})}_{\text{in } W} \in g_{i_2} W$$

$$\Theta(s_1) g_{i_2}(r(h_2)\underline{w}) \underset{\substack{\uparrow \\ s_1 g_{i_2} = g_{i_1} h_1}}{=} g_{i_1}(r(h_1) r(h_2) \underline{w}) = g_{i_1}(\underbrace{r(h_1 h_2)\underline{w}}_{\text{in } W})$$

$$\Theta(s_1 s_2) g_i \underline{w} \underset{\substack{\uparrow \\ s_1 s_2 g_i = g_{i_1}(h_1 h_2)}}{=} g_{i_1} r(h_1 h_2)\underline{w}.$$

So $\Theta(s_1 s_2) g_i \underline{w} = \Theta(s_1)\, \Theta(s_2) g_i \underline{w}$, $\forall \underline{w} \in W$, $\forall s_1, s_2 \in G$, $\forall i = 1 .. K$. ✓

This shows that (Θ, V) is a well defined representation of G. Clearly V has dimension

$$\dim V = K \dim W = \frac{|G|}{|H|} \dim r$$

and the set $\{ g_j \underline{w}_\ell \}_{\substack{j = 1..K = |G|/|H| \\ \ell = 1..n = \dim W}}$ are a basis of $V = \bigoplus_{i=1}^{K} g_i W$. Next, we show that (Θ, V) is equivalent to (ρ, X)

EQUIVALENCE OF THE TWO DEFINITIONS

- we define a map $T : X \to V$ and we show that
  1) T is an isomorphism
  2) T is an intertwining operator, i.e. the diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\rho(s)"'] & V \arrow[d, "\Theta(s)"] \\ X \arrow[r, "T"] & V \end{tikzcd}$$

commutes for all $s \in G$.

We know that $\{f_{j\ell}\}_{\substack{j=1..k=|G|/|H| \\ \ell=1..n=\dim W}}$ and $\{g_j \underline{w}_\ell\}_{\substack{j=1..k \\ \ell=1..n}}$

- are basis of X and V respectively.

Define $T : X \to V$ by $T(f_{j\ell}) = g_j \underline{w}_\ell$, $\forall j = 1..k$, $\forall \ell = 1..n$.

Clearly T is a vector space isomorphism.

Next, we show that

$$\Theta(s)\, T(f_{j\ell}) = T \rho(s)(f_{j\ell})$$

$\forall s \in G$, $\forall j = 1..k$, $\forall \ell = 1..n$.

LEFT HAND SIDE:

- $\Theta(s)\, T(f_{j\ell}) = \Theta(s)(g_j \underline{w}_\ell) = g_{j'}\, t(h)\, \underline{w}_\ell =$

  suppose that $s g_j = g_{j'} h$; suppose that $t(h)\underline{w}_\ell = \sum_{m=1}^{n} a_m \underline{w}_m$

- $= \sum_{m=1}^{n} a_m\, g_{j'} \underline{w}_m$

RIGHT HAND SIDE

- $T\rho(s) f_{je} = ?$

First, we need to determine $\rho(s) f_{je}$.
It's enough to understand $(\rho(s) f_{je})(g_r)$ $\forall r = 1 \dots k$.

$$\rho(s) f_{je}(g_r) = f_{je}(s^{-1} g_r) = \begin{cases} 0 & \text{if } s^{-1} g_r \notin g_j H \\ f_{je}(g_j \tilde{h}) & \text{if } s^{-1} g_r = g_j \tilde{h}. \end{cases}$$

remember That f_{je} is supported on $g_j H$.

Notice That $s^{-1} g_r \in g_j H \iff g_r \in s g_j H \iff r = j'$.

$s g_j = g_{j'} h$

Moreover, $s g_j = g_{j'} h \Rightarrow s^{-1} g_{j'} h = g_j \Rightarrow s^{-1} g_{j'} = g_j h^{-1}$.
In The previous notations, $\tilde{h} = h^{-1}$.

So $\rho(s) f_{je}(g_r) = 0$ $\forall r \neq j'$ and

$$\rho(s) f_{je}(g_{j'}) = f_{je}(g_j h^{-1}) = t((h^{-1})^{-1}) \underline{w}_e =$$

$$= t(h) \underline{w}_e = \sum_{m=1}^{n} a_m \underline{w}_m.$$

We can write:

$$\rho(s) f_{je} = \sum_{m=1}^{n} a_m f_{j'm}.$$

- Next, we apply T To iT:

☺

$$T\rho(s) f_{je} = \sum_{m=1}^{n} a_m T(f_{j'm}) = \sum_{m=1}^{n} a_m g_{j'} \underline{w}_m = \theta(s) T f_{je}$$

We have proved that

$$T\rho(s) f_{je} = \theta(s) T f_{je} \quad \forall j = 1 \ldots K = \frac{|G|}{|H|}, \ \forall \ell = 1 \ldots n = \dim W.$$

So $T\rho(s) = \theta(s) T$, $\forall s \in G$ and the isomorphism T is an intertwining operator.

$\Rightarrow$ the representations (ρ, X) and (θ, V) of G are equivalent.

$\Longrightarrow$ the two definitions of induced representations are equivalent!

Remark: For practical purposes, it's more convenient to use the second definition (i.e. (θ, V)), although it's less natural...

Remark: We should also prove that our definitions of (θ, V) and (ρ, X) are independent of the choice of the coset representatives $\{g_1, \ldots, g_K\}$.
We omit the proof, for brevity reasons.

EXAMPLES OF INDUCED REPRESENTATIONS

EXAMPLE #1 Let $G = S_3$, let $H = \{1, (23)\}$ and let $(\Gamma, \mathbb{C})$ be the trivial representation of H.

We choose 1, (12) and (13) are representatives in G for the left cosets of H:

$$G = 1H \uplus (12)H \uplus (13)H$$

with

$$1H = \{1, (23)\}$$

$$(12)H = \{(12), (123)\}$$

$$(13)H = \{(13), (132)\}.$$

We construct the induced representation using the 2^{nd} definition

In our notations:

$$W = \mathbb{C}; \quad K = 3 = \frac{|G|}{|H|}; \quad g_1 = 1; \; g_2 = (12); \; g_3 = (13)$$

and $V = 1\mathbb{C} \oplus (12)\mathbb{C} \oplus (13)\mathbb{C}$.

We write an element of V as

$$\underline{v} = 1x + (12)y + (13)z,$$

and we study the action of $\sigma \in G$ on $\underline{v}$.

If $\sigma = (12)$, then

$$\theta((12))\,\underline{v} = \underbrace{\theta(12)\,1}_{(12)1 = \underset{g_2}{(12)}\,\underset{\text{in}H}{1}}\, x + \underbrace{\theta(12)(12)}_{(12)(12) = \underset{g_1}{1}\cdot\underset{\text{in}H}{1}}\, y + \underbrace{\theta(12)(13)}_{(12)(13) = (132) = \underset{g_3}{(13)}\,\underset{\text{in}H}{(23)}}\, z =$$

$$= (12)\,t(1)\,x + 1\,t(1)\,y + (13)\,t(23)\,z =$$

$$\underset{\substack{\uparrow \\ \text{tr is the} \\ \text{Trivial representation}}}{=} (12)x + 1y + (13)z = 1y + (12)x + (13)z$$

$\Rightarrow$ The matrix of $\Theta(12)$ wrt the basis $\{1\cdot 1, (12)\cdot 1, (13)\cdot 1\}$ is

$$\Theta(12) \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ Notice that $\chi_\Theta(12) = 1$.

If $\sigma = (123)$, then

$$\Theta(123)v = \underbrace{\Theta(123)1}_{\substack{(123)\cdot 1 = (12)\cdot(23) \\ \uparrow g_2 \quad \uparrow \text{in } H}} x + \underbrace{\Theta(123)(12)}_{\substack{(123)(12) = (13) = \\ = (13)\cdot 1 \\ \uparrow g_3 \quad \uparrow \text{in } H}} y + \underbrace{\Theta(123)(13)}_{\substack{(123)(13) = (23) = \\ = 1\cdot(23) \\ \uparrow g_1 \quad \uparrow \text{in } H}} z =$$

$$= (12)\, tr(23)x + (13)\, tr(1)y + 1\, tr(23)z =$$

$$\underset{\substack{\uparrow \\ tr = \text{Trivial}}}{=} (12)x + (13)y + 1z = 1z + (12)x + (13)y$$

$$\Rightarrow \quad \Theta(123) \rightsquigarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ and $\chi_\Theta(123) = 0$.

It follows that $\Theta = \mathrm{Ind}_H^G(tr)$ has character

1	(12)	(123)
3	1	0

.

A quick glance at The character Table of $G = S_3$

	1	(12)	(123)
U	1	1	1
U'	1	−1	1
V	2	0	−1

Shows That $\Theta = U \oplus V$.

Similarly, if we induced t' = sign representation of H we find :

$$\Theta'(12) = (12)\,t'(1)\,x + 1\,t'(1)\,y + (13)\,t'(23)\,z =$$
$$= (12)x + 1y - (13)z =$$
$$= 1y + (12)x - (13)z$$

$$\Rightarrow \Theta'(12) \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ and $\chi_{\Theta'}(12) = -1$;

and

$$\Theta'(123) = (12)\,t'(23)\,x + (13)\,t'(1)\,y + 1\,t'(23)\,z =$$
$$= -(12)x + (13)y - 1z =$$
$$= -1z - (12)x + (13)y$$

$$\Rightarrow \Theta'(123) \rightsquigarrow \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ and $\chi_{\Theta'}(123) = 0$.

The character of Θ' is

1	(12)	(123)
3	−1	0

, hence

$\Theta' = U' \oplus V$.

Example #2

- $G = D_6 = \langle a, b : a^3 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle$
  and $H = \langle a \rangle$ (order 3, index 2).

The group H has 3 one-dimensional representations t_1, t_2, t_3 given by:

	1	a	a^2
t_1	1	1	1
t_2	1	ω	$\omega^2 = \bar{\omega}$
t_3	1	$\omega^2 = \bar{\omega}$	ω

where $\omega = e^{2\pi i/3}$.

- Let's induce each of these representations to G.
  In every case, $W = \mathbb{C}$ and $V_i = V = 1\,\mathbb{C} \oplus b\,\mathbb{C}$.
  [We can choose $\{1, b\}$ as representatives in G for the left cosets of G: $G = 1H \uplus bH$, and
  $1H = \{a, a^2, 1\}$
  $bH = \{ba = a^3 b,\ ba^2 = ab,\ b\}$.]

Also, for every $i = 1, 2, 3$, we can write:

- $\theta_i(a)(1x + by) = \underbrace{\theta_i(a)1}_{\substack{a\cdot 1 = 1\cdot a \\ g_1 \quad \text{in } H}} x + \underbrace{\theta_i(a)b}_{\substack{ab = ba^2 \\ g_2 \quad \text{in } H}} y =$

$= 1\, t_i(a) x + b\, t_i(a^2) y.$

• $\theta_i(b)(1x+by) = \underbrace{\theta_i(b)1}_{\substack{b\cdot 1 = b\cdot 1 \\ g_2 \quad \text{in } H}} x + \underbrace{\theta_i(b)b}_{\substack{bb = 1 = 1\cdot 1 \\ g_1 \quad \text{in } H}} y =$

$= b\,t_i(1)\,x + 1\,t_i(1)\,y \underset{t_i(1)=1 \ \forall i=1..3}{=} 1\cdot y + b\cdot x$

The matrices of $\theta_i(a)$ and $\theta_i(b)$ w.r.t. the basis $\{1\cdot 1, b\cdot 1\}$ of V are:

$$\theta_i(a) \rightsquigarrow \begin{bmatrix} t_i(a) & 0 \\ 0 & t_i(a^2) \end{bmatrix}$$

$$\theta_i(b) \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

When $i=1$, $\theta_1 = \mathrm{Ind}_H^G(\text{trivial})$ and $\begin{cases} \theta_1(a) = \mathrm{Id} \\ \theta_1(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{cases}$

When $i=2$, $\theta_2 = \mathrm{Ind}_H^G(t_2)$ and $\begin{cases} \theta_1(a) = \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix} \\ \theta_2(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}$

When $i=3$, $\theta_3 = \mathrm{Ind}_H^G(t_3)$ and $\begin{cases} \theta_1(a) = \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} \\ \theta_2(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{cases}$

Compare these results with the character table of D_6:

	1	cc(a)	cc(b)
ρ_1	1	1	1
ρ_2	1	1	-1
ρ_3	2	-1	0

Then it's clear that

$$\begin{cases} \theta_1 = \rho_1 \oplus \rho_2 \\ \theta_2 = \rho_3 \\ \theta_3 = \rho_3. \end{cases}$$

Example #3

Let Γ be the regular representation of H, so $W = \bigoplus_{h \in H} \mathbb{C} e_h$.

Choose $g_1, \dots, g_k \in G$ st $G = \bigoplus_{i=1}^{k} g_i H$.

Let $(\theta, V) = \mathrm{Ind}_H^G (\Gamma, W)$.

then $V = \bigoplus_{i=1}^{k} g_i W = \bigoplus_{\substack{i=1..k \\ h \in H}} \mathbb{C} g_i e_h$.

An element $s \in G$ acts on $g_i e_h$ by:

$$s(g_i e_h) \underset{\substack{\uparrow \\ s g_i = g_i' h'}}{=} g_i' \Gamma(h') e_h = g_i' e_{h'h}.$$

Claim: (θ, V) is equivalent to the regular representation of G.

Denote the reg. rep. of G by (ρ, X). then $X = \bigoplus_{g \in G} \mathbb{C} e_g$ and $\rho(s) e_g = e_{sg}$ $\forall s, g \in G$.

We define an isomorphism $T: V \to X$ by

$$T(g_i e_h) = e_{g_i h}, \ \forall i = 1..k, \forall h \in H.$$

Because $G = \bigoplus_{i=1}^{k} g_i H$, T carries a basis of V into a basis of X, so it's invertible. Moreover T is an intertwining operator:

- $\rho(s)\, T(g_i e_h) = \rho(s)(e_{g_i h}) = e_{s g_i h}$
- $T\,\theta(s)(g_i e_h) \underset{\substack{\uparrow \\ s g_i = g_{i'} h'}}{=} T(g_{i'}\, e_{h' h}) = e_{\underbrace{g_{i'} h'}_{s g_i} h} = e_{s g_i h}$ ✓

$[\forall s \in G, \forall h \in H, \forall i = 1 \ldots k]$.

$\Rightarrow \mathrm{Ind}_H^G$ (regular repr. of H) $=$ regular representation of G.

EXAMPLE # 4. As usual, let $H \leq G$, and let $G = \biguplus_{i=1}^{k} g_i H$.

$\mathrm{Ind}_H^G(\text{trivial})$ = permutation representation of G associated with the natural action of G on the set $\{g_1 H, g_2 H, \ldots, g_k H\}$.

[if $s \in G$, and $(s g_i)H = g_{i'} H$, then define $s \cdot [g_i H] = [g_{i'} H]$.]

proof – $V = \mathrm{Ind}_H^G \text{triv.} = \bigoplus_{i=1}^{k} g_i \mathbb{C}$; $\theta(s)\, g_i a \underset{\substack{\uparrow \\ s g_i = g_{i'} h}}{=} g_{i'} \underbrace{\Gamma(h) a}_{= a \text{ bc } \Gamma = \text{Trivial}} = g_{i'} a$.

Now consider the permutation representation associated to the action of G on cosets:

$U = \bigoplus_{i=1}^{k} \mathbb{C} e_{g_i H}$; $\rho(s)\, e_{g_i H} \underset{\substack{\uparrow \\ s \cdot g_i H = g_{i'} H \\ \text{if } s g_i = g_{i'} h}}{=} e_{s \cdot g_i H} = e_{g_{i'} H}$.

It's clear that the isomorphism $T: V \to U$, $g_i 1 \mapsto e_{g_i H}$ is an intertwining operator.