Restriction to a subgroup

In this lecture we study the restriction of a representation of G to a subgroup $H \leq G$. In particular, we focus on the case in which H is a normal subgroup of G of index 2.

Introduction

Let G be a finite group and let $H \leq G$ be a subgroup. Every representation ρ of G in V restricts to a representation ρ_H of H on the same space V:

$$\rho_H = \mathrm{Res}^G_H \rho : H \longrightarrow GL(V), \; h \mapsto \rho(h).$$

We call ρ_H the restriction of ρ from G to H.
For every $h \in H$, $\chi_{\rho_H}(h) = \mathrm{trace}(\rho(h)) = \chi_\rho(h)$. So the character of ρ_H is obtained from the character of ρ by restriction to H.

<u>Remark</u> Let ρ be an irreducible representation of G. The restriction ρ_H of ρ to H is not necessarily irreducible.

<u>A trivial example</u>: Suppose that ρ is an irreducible representation of G of degree $\geqslant 2$, and that H is an abelian subgroup of G. The restriction of ρ to H is necessarily reducible (because every irreducible representation of an abelian group has degree 1).

<u>A less trivial example</u>: Let $G = S_4$ and $H = S_3$ (we realize S_3 as a subgroup of S_4 in the obvious way: S_3 consists of

all the permutations of $\{1,2,3,4\}$ that fix 4).
Recall the character table of S_4:

S_4	1	(12)	(123)	(1234)	(12)(34)
ρ_1	1	1	1	1	1
ρ_2	1	-1	1	-1	1
ρ_3	3	1	0	-1	-1
ρ_4	3	-1	0	1	-1
ρ_5	2	0	-1	0	2

Restriction to S_3 gives:

S_3	1	(12)	(123)
$\mathrm{Res}_{S_3}^{S_4}(\rho_1)$	1	1	1
$\mathrm{Res}_{S_3}^{S_4}(\rho_2)$	1	-1	1
$\mathrm{Res}_{S_3}^{S_4}(\rho_3)$	3	1	0
$\mathrm{Res}_{S_3}^{S_4}(\rho_4)$	3	-1	0
$\mathrm{Res}_{S_3}^{S_4}(\rho_5)$	2	0	-1

A comparison with the character table of S_3

S_3	1	(12)	(123)
U	1	1	1
U'	1	-1	1
V	2	0	-1

shows that only ρ_1, ρ_2 and ρ_5 remain irreducible when restricted to S_3. Moreover

- $\mathrm{Res}_{S_3}^{S_4}(\rho_3) = U \oplus V$
- $\mathrm{Res}_{S_3}^{S_4}(\rho_4) = U' \oplus V$.

General problem: Given an irreducible representation ρ of G and given a subgroup H of G, understand whether $\rho_H = \mathrm{Res}^G_H \rho$ is irreducible. If ρ_H is reducible, find a decomposition of ρ_H as a direct sum of isotypic components of irreducible representations of H.

$$\rho_H = (\mu_1^{\oplus d_1}) \oplus (\mu_2^{\oplus d_2}) \oplus \dots \oplus (\mu_k^{\oplus d_k}).$$

Notations - If μ_i is an irreducible representation of H and $d_i > 0$, we say that μ_i is a constituent of $\rho_H = \mathrm{Res}^G_H \rho$, and that d_i is the multiplicity of μ_i in ρ_H.

Recall that $d_i = \dim_{\mathbb{C}}[\mathrm{Hom}_H(\mu_i, \rho_H)] = \langle \chi_{\mu_i}, \chi_{\rho_H} \rangle =$

$$= \frac{1}{|H|} \sum_{h \in H} \overline{\chi_{\mu_i}(h)}\, \chi_\rho(h).$$

The following proposition shows that every irreducible representation of H is a constituent of some irreducible representation of G.

Proposition - Let μ be an irreducible representation of H. Then there exists an irreducible representation ρ of G s.t. $\langle \chi_\mu, \chi_{\mathrm{Res}^G_H \rho} \rangle > 0$.

Proof - Because the regular representation of G contains every irreducible (with multiplicity equal to its dimension) and because

$$\langle \chi_\mu, \chi_{\mathrm{Res}^G_H(\mathrm{REGULAR})} \rangle = \langle \chi_\mu, \sum_{\substack{\text{all irred.}\\ \text{reprs of } G}} (\dim \rho)\, \chi_{\mathrm{Res}^G_H(\rho)} \rangle =$$

$$= \sum_{\substack{\text{all irred.}\\ \text{reprs of } G}} \underbrace{(\dim \rho)}_{>0} \underbrace{\langle \chi_\mu, \chi_{\mathrm{Res}^G_H(\rho)} \rangle}_{\geq 0}$$

it is clearly enough to show that $\langle \chi_\mu, \chi_{\mathrm{Res}^G_H(\mathrm{REGULAR})} \rangle$ is strictly greater than zero, i.e. μ is a constituent of the restriction to H of the regular representation of G.

this is easy to do:

$$\langle \chi_\mu, \chi_{\mathrm{Res}^G_H(\mathrm{REGULAR})} \rangle = \frac{1}{|H|} \sum_{h \in H} \overline{\chi_\mu(h)}\, \chi_{\substack{\text{REGULAR}\\ \text{REP. OF } G}}(h) =$$

$$= \frac{1}{|H|} \chi_\mu(e_G)\, |G| = (\text{degree of } \mu) \cdot [G:H] > 0.$$

$\square$

Recall that
$$\chi_{\text{REG. OF } G}(g) = \begin{cases} |G| & \text{if } g = e_G \\ 0 & \text{o.w.} \end{cases}$$

So if we look inside the restriction to H of the various irreducible representations of G, we find ALL the irreducible representations of H !!!

NEXT QUESTION: how $\underbrace{\quad}_{\text{can we estimate the number of}}$ constituents of the restriction to H of an irreducible representation of G?

Proposition - Let ρ be an irreducible representation of G, and let

$$\rho_H = (\mu_1^{\oplus d_1}) \oplus \cdots \oplus (\mu_k^{\oplus d_k})$$

be the decomposition of $\rho_H = \mathrm{Res}_H^G(\rho)$ into irreducible representations of H. then

$$\sum_{i=1}^{k} d_i^2 \leq [G:H].$$

Proof - We first show that

$$\sum_{i=1}^{k} d_i^2 = \langle \chi_{\rho_H}, \chi_{\rho_H} \rangle$$

and then prove that $\langle \chi_{\rho_H}, \chi_{\rho_H} \rangle \leq [G:H]$.

STEP 1 $\langle \chi_{\rho_H}, \chi_{\rho_H} \rangle \underset{\uparrow \atop \rho_H = \bigoplus_{i=1}^{k} (\mu_i^{\oplus d_i})}{=} \langle \sum_{i=1}^{k} d_i \chi_{\mu_i}, \sum_{j=1}^{k} d_j \chi_{\mu_j} \rangle =$

$$= \sum_{i,j=1\ldots k} d_i d_j \langle \chi_{\mu_i}, \chi_{\mu_j} \rangle \underset{\uparrow}{=} \sum_{i=1}^{k} d_i^2. \checkmark$$

$\langle \chi_{\mu_i}, \chi_{\mu_j} \rangle = \delta_{ij}$
because the $\{\mu_j\text{'s}\}$ are distinct irreducible ...

STEP 2 $\langle \chi_{\rho_H}, \chi_{\rho_H} \rangle = \frac{1}{|H|} \sum_{h \in H} \overline{\chi_{\rho_H}(h)}\, \chi_{\rho_H}(h) =$

$$= \frac{1}{|H|} \sum_{h\in H} |\chi_{\rho_H}(h)|^2 = \frac{1}{|H|} \sum_{h \in H} |\chi_\rho(h)|^2 \leq \frac{1}{|H|} \sum_{g \in G} |\chi_\rho(g)|^2 =$$

$$= \frac{|G|}{|H|} \langle \chi_\rho, \chi_\rho \rangle \underset{\uparrow}{=} \frac{|G|}{|H|} \cdot 1 = \frac{|G|}{|H|} = [G:H]. \checkmark$$

ρ is an irreducible representation of G

This complete the proo

$\square$

When H is a <u>normal</u> subgroup of G, we can say more: every constituent of ρ_H has the same dimension!

<u>Theorem</u> - If $H \trianglelefteq G$, then for every irreducible representation (ρ,V) of G there exists an irreducible representation W of H s.t. $\rho_H = \sum_{g\in G} \rho(g)W$.

<u>proof</u> - Suppose that W is a subspace of V stable under H. Because H is normal, the subspace $\rho(g)W$ is also H-stable, $\forall g\in G$. Indeed $\forall w\in W$ we can write:

$$\rho(h)\rho(g)w = \rho(g)\rho(\underbrace{g^{-1}hg}_{h_1\in H})w = \rho(g)\underbrace{\rho(h_1)w}_{w_1\in W} = \rho(g)w_1.$$

Consider the subspace

$$U = \sum_{g\in G} \rho(g)W \subseteq V.$$

[The sum is not a direct sum, for instance $\rho(h)W = W$ $\forall h\in W$. In general, $\rho(g_1)W$ migth be equal to $\rho(g_2)W$ even if $g_1 \neq g_2$.]

U is stable under H (because each $\rho(g)W$ is H-stable) but it's also stable under G:

$$\rho(g)\rho(g_1)w = \rho(gg_1)w \in \sum_{\tilde{g}\in G} \rho(\tilde{g})W, \ \forall w\in W, \forall g, g_1 \in G$$

But V is an irreducible representation of G and U is non-zero, so it follows that $V = U$.

$\sim V = \sum \rho(g)W.$

Notice that, as vector spaces, all the subspaces $\rho(g)W$ are isomorphic to W. In particular, they all have the same dimension.

Choose representatives $g_1, g_2, \dots, g_r$ s.t.

$$V = \sum_{g \in G} \rho(g) W = \sum_{i=1}^{r} \rho(g_i) W$$

and $\rho(g_i)W \neq \rho(g_j)W \quad \forall i,j = 1..r$.

Then $V = \bigoplus_{i=1}^{r} \rho(g_i)W$. [Two subspaces of the same dimension are equal or they have zero intersection].

For all $i = 1..r$, $\rho(g_i)W$ is a representation of H.

Notice that, as an H-representation, $\rho(g_i)H$ needs <u>not</u> to be isomorphic to $\rho(g_j)W$ if $i \neq j$.

Still, $\rho(g_i)W$ and $\rho(g_j)W$ have the same dimension

$\Rightarrow \rho_H$ decomposes as a direct sum of irreducible representations of the same dimension.

<u>EXAMPLE</u> Suppose that $H \trianglelefteq G$, and that ρ is an irreducible representation of G of dimension p, with p a prime. Then either ρ_H is irreducible, or ρ_H is the sum of p linear characters of H.

[Indeed, if $\rho_H = \bigoplus_{i=1}^{k} \rho(g_i)W$, then $p = \dim(\rho) = k \dim W$, so both k and $\dim(W)$ must divide p.

Notice that the latter can occur only if $p \leq [G:H]$.

<u>Corollary</u> If H is a <u>normal</u> subgroup of <u>index 2</u>, ~~then~~ and ρ is any ~~for every~~ irreducible representation ~~ρ~~ of G, the restriction ρ_H of ρ to H is either irreducible or the direct sum of two irreducible inequivalent representations of H of the same dimension.

[Indeed, if $\sum_{i=1}^{k} d_i^2 \leq [G:H] = 2$, then either $k=1$ and $d_1=1$ (ie ρ_H is irreducible) or $k=2$ and $d_1=d_2=1$ (i.e. ρ_H is of the form $\rho_H = W_1 \oplus W_2$, with $W_1 \not\cong W_2$. Notice that $\dim W_1 = \dim W_2$ by the previous theorem].

From now on, H will always be a (normal) subgroup of G of index 2.

<u>Remark</u>: If $H \trianglelefteq G$ has index 2, every irreducible representation of G of <u>odd</u> dimension has an irreducible restriction to H.

<u>Question</u>: How can we predict if the restriction of an even dimensional representation is reducible or irreducible?

We will show that this question has a really simple answer: if the character of ρ is identically zero on $G-H$, then ρ_H is reducible. If there exists a $g \in G$, with $g \notin H$, s.t. $\chi_\rho(g) \neq 0$, then ρ_H is irreducible.

Before proving this theorem, we need to introduce some notations. Because H is a normal subgroup of index 2, the quotient group G/H is a well defined abelian group of order 2, and it has 2 characters: $\tau_1 : G/H \to \mathbb{C}^*$ trivial
$\tau_2 : G/H \to \mathbb{C}^*$ with $\tau_2(gH) = -1$ if g

We can lift t_2 to a character of G (by composing with the projection $G \to G/H$). The result is a non trivial linear character of G, that we denote by λ, which is trivial on H and has the value -1 on $G-H$:

$$\lambda(g) = \begin{cases} 1 & \text{if } g \in H \\ -1 & \text{if } g \notin H. \end{cases}$$

Let χ_ρ be the character of ρ. Because χ_ρ is irreducible and λ is linear, $\chi_\rho \cdot \lambda$ is also an irreducible character of G. There are two possibilities:

① $\chi_\rho \cdot \lambda = \chi_\rho \iff \chi_\rho\big|_{G-H} \equiv 0$

② $\chi_\rho \cdot \lambda \neq \chi_\rho \iff \exists g \in G,\ g \notin H$ ST $\chi_\rho(g) \neq 0$.

The claim is that ρ_H is reducible if ① occurs, and irreducible if ② occurs.

<u>Proposition</u> Let $H \trianglelefteq G$ be a normal subgroup of G of index 2. Let ρ be an irreducible representation of G. The following two conditions are equivalent:

(i) $\mathrm{Res}_H^G \rho = \rho_H$ is reducible, i.e. $\langle \chi_{\rho_H}, \chi_{\rho_H} \rangle \geq 2$

(ii) $\chi_\rho \cdot \lambda = \chi_\rho$, i.e. $\forall g \in G-H$: $\chi_\rho(g) = 0$.

<u>Proof</u> - To test the reducibility of ρ_H we compute the inner product $\langle \chi_{\rho_H}, \chi_{\rho_H} \rangle$:

$$\langle \chi_{\rho_H}, \chi_{\rho_H} \rangle = \frac{1}{|H|} \sum_{h \in G} \overline{\chi_{\rho_H}(h)}\, \chi_{\rho_H}(h) = \frac{1}{|H|} \sum_{h \in G} \overline{\chi_\rho(h)}\, \chi_\rho(h) =$$

$$= \frac{1}{|H|}\sum_{h\in G}|\chi_\rho(h)|^2 = \underbrace{\frac{|G|}{|H|}}_{[G:H]=2}\left(\underbrace{\frac{1}{|G|}\sum_{g\in G}|\chi_\rho(g)|^2}_{\langle\chi_\rho,\chi_\rho\rangle=1} - \frac{1}{|G|}\sum_{g\in G-H}|\chi_\rho(g)|^2\right) =$$

$$= 2\left[1 - \frac{1}{|G|}\sum_{g\in G-H}|\chi_\rho(g)|^2\right] = 2 - \underbrace{\boxed{\frac{2}{|G|}\sum_{g\in G-H}|\chi_\rho(g)|^2}}_{\geq 0}$$

So, because ρ_H is reducible if and only if $\langle\chi_{\rho_H},\chi_{\rho_H}\rangle \geq 2$ we see that

$$\boxed{\rho_H \text{ reducible}} \Longleftrightarrow \boxed{\frac{1}{|H|}\sum_{g\in G-H}|\chi_\rho(g)|^2 = 0} \Longleftrightarrow \boxed{\chi|_{G-H} \equiv 0}$$

$\square$

This proposition shows that, when $H \trianglelefteq G$ of index 2, we can predict from the character table which irreducible characters of G will restrict irreducibly to H.

As an example, we look at the character table of A_4.

Set $G = S_4$; $H = A_4$.
[A_4 has index 2 in S_4, so the theory applies...]

$\longrightarrow$

	H	G-H	H	G-H	H
S_4	1	(12)	(123)	(1234)	(12)(34)
ρ_1	1	1	1	1	1
ρ_2	1	-1	1	-1	1
ρ_3	3	1	0	-1	-1
ρ_4	3	-1	0	1	-1
ρ_5	2	0	-1	0	2

understand which conjugacy classes have a representative in H

focus the attention on these 2 columns (corresponding to conjugacy classes in G-H).

Recall that ρ_H is irreducible $\iff \chi_\rho\big|_{G-H} \neq 0$.

We immediately tell that $\rho_1, \rho_2, \rho_3, \rho_4$ have an irreducible restriction to $H = A_4$, while $\rho_5\big|_{A_4}$ spits into the direct sum of two inequivalent irreducible representation of A_4 (of equal dimension): $\rho_5\big|_{A_5} = \alpha + \beta$, $\dim\alpha = \dim\beta = 1$.

the character table of A_4 is therefore given by:

notice that (123) does not commute with any odd permutation. So the S_4-conjugacy class of (123) splits into 2 A_4-conjugacy classes

A_4	1	(123)	(132)	(12)(34)
Res ρ_1	1	1	1	1
Res ρ_2	1	1	1	1
Res ρ_3	3	0	0	-1
Res ρ_4	3	0	0	-1
α	1	?	?	?
β	1	?	?	?

constituents of Res(ρ_5)

We know that

A_4	1	(123)	(132)	(12)(34)
α	1	?	?	?
β	1	?	?	?
$\alpha+\beta$	2	-1	-1	2

How can we guess the missing entries?
α and β are linear characters, (12)(34) has order 2, so $\alpha((12)(34))$ and $\beta((12)(34))$ must be +1 or -1. Because their sum is equal to 2, they must both be equal to +1.

(123) has order 3, so $\alpha(123)$ and $\beta(123)$ must be cubic roots of unity: 1, $\omega = e^{2\pi i/3}$ and $\bar{\omega} = e^{-2\pi i/3}$. $\alpha(123) + \beta(123)$ must be equal to -1. So, without loss of generality, we can assume that

$$\alpha(123) = \omega$$
$$\beta(123) = \bar{\omega}.$$

This gives: $\alpha(132) = \overline{\alpha(123)} = \bar{\omega}$
$\beta(132) = \overline{\beta(123)} = \omega.$

The character table of A_4 is therefore given by:

A_4	1	(123)	(132)	(12)(34)
Res ρ_1	1	1	1	1
Res ρ_3	3	0	0	-1
α	1	ω	$\bar{\omega}$	1
β	1	$\bar{\omega}$	ω	1

constituents of Res(ρ_5)

This example shows that

- every irreducible representation of $H = A_4$ is a constituent of some irreducible representation of $G = S_4$

- if ρ is an irred. representation of G st ρ_H is irreducible, then there exists exactly <u>one other</u> irreducible representation ρ' of G st $\mathrm{Res}^G_H(\rho) = \mathrm{Res}^G_H(\rho')$.

- if ρ is an irred. representation of G st ρ_H is reducible, then there exists <u>no other</u> irred. repres. ρ' of G st $\mathrm{Res}^G_H(\rho) = \mathrm{Res}^G_H(\rho')$.

All of these properties generalize to any (normal) subgroup H of index 2 of any group G.

<u>Theorem</u> Let $H \trianglelefteq G$ of index 2. Let ρ be an irreducible representation of G. Then

① If ρ_H is irreducible ($\Leftrightarrow \chi_\rho \neq \lambda \cdot \chi_\rho$) then

$\mathrm{Res}^G_H \phi = \rho_H \left(= \mathrm{Res}^G_H \rho\right)$ if and only if $\chi_\phi = \chi_\rho$ or $\chi_\rho \cdot \lambda$.

[So there is exactly one other irreducible repr. of G that has the same restriction to H].

② If ρ_H is reducible ($\Leftrightarrow \chi_\rho = \lambda \cdot \chi_\rho$), and $\rho_H = \varphi_1 \oplus \varphi_2$, and Φ is an irreducible representation of G having

φ_1 and/or φ_2 as constituents, then $\Phi = \rho$.

proof - ① Assume that ρ_H is irreducible. Then $\chi_\rho \neq \lambda\cdot\chi_\rho$ -

Because

$$\lambda\cdot\chi_\rho(g) = \begin{cases} \chi_\rho(g) & \text{if } g\in H \\ -\chi_\rho(g) & \text{if } g\notin H \end{cases}$$

it's clear that $\lambda\cdot\chi_\rho|_H = \chi_\rho|_H$.

We need to show that if $\Phi_H = \rho_H$ for some irred. representation of G, then $\chi_\phi = \chi_\rho$ or $\chi_\phi = \lambda\cdot\chi_\rho$ -

Because all of the representations involved are irreducible, this is equivalent to proving that

$$\langle \chi_\phi, \chi_\rho + \lambda\cdot\chi_\rho \rangle = 1.$$

We can do this easily:

$$\langle \chi_\phi, \chi_\rho + \lambda\cdot\chi_\rho \rangle = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_\phi(g)}\,\underbrace{[\chi_\rho(g) + (\lambda\cdot\chi_\rho)(g)]}_{\substack{0 \text{ if } g\in G-H \\ \text{and} \\ 2\chi_\rho(g) \text{ if } g\in H}} =$$

$$= \frac{1}{|G|}\sum_{g\in H} \overline{\chi_\phi(g)}\, 2\chi_\rho(g) =$$

$$= \frac{|H|}{|G|}\frac{1}{|H|}\, 2\sum_{h\in H} \overline{\chi_{\phi_H}(h)}\,\chi_{\rho_H}(h) =$$

$\frac{|H|}{|G|} = \frac{1}{2}$

$$= \langle \chi_{\phi_H}, \chi_{\rho_H} \rangle = 1$$

because ρ_H is irreducible and $\phi_H = \rho_H$.

This completes the proof of part ①.

② Now assume that ρ_H is reducible ($\Leftrightarrow \chi_\rho = \lambda \cdot \chi_\rho^{\omega}$ because $\chi_\rho|_{G-H} \equiv 0$). Suppose that $\rho_H = \varphi_1 \oplus \varphi_2$ and that ϕ is some irreducible representation of G s.t. ϕ_H contains φ_1 and/or φ_2.

We want to show that $\rho = \phi$. It's of course sufficient to prove that $\langle \chi_\rho, \chi_\phi \rangle > 0$.

By a direct computation:

$$\langle \chi_\phi, \chi_\rho \rangle = \frac{1}{|G|} \sum_{g\in G} \overline{\chi_\phi(g)}\, \chi_\rho(g) \underset{\chi_\rho|_{G-H}\equiv 0}{=}$$

$$= \frac{1}{|G|} \sum_{g\in H} \overline{\chi_\phi(g)}\, \chi_\rho(g) =$$

$$= \frac{|H|}{|G|} \frac{1}{|H|} \sum_{h\in H} \overline{\chi_{\phi_H}(h)}\, \chi_{\rho_H}(h) = \frac{1}{2} \langle \chi_{\phi_H}, \chi_{\varphi_1} + \chi_{\varphi_2} \rangle =$$

$$= \frac{1}{2} \Big[\langle \chi_{\phi_H}, \chi_{\varphi_1} \rangle + \langle \chi_{\phi_H}, \chi_{\varphi_2} \rangle \Big] > 0.$$

at least one of them is >0
because φ_1 and/or φ_2 is a constituent of ϕ_H

This completes the proof. □

I) We conclude this lecture on restricted representations with one final problem:

$\longrightarrow$

<u>Problem</u> – It is known that the complete list of degrees of the irreducible characters of S_7 is:

$$1, 1, 6, 6, 14, 14, 14, 14, 15, 15, 20, 21, 21, 35, 35.$$

Also, we know that A_7 has exactly 9 conjugacy classes. Find the complete list of degrees of the irreducible characters of A_7.

▶ Let ρ be one of the two irreducible representation of S_7 of degree 1, or 15 or 21 or 35. Because ρ has odd dimension, ρ_{A_7} must be irreducible.

Notice that the other representation of S_7 of degree 1, 15, 21 or 35 gives rise to the same restriction to S_7 (the character is of the form $\lambda \cdot \chi_\rho \ldots$).

We obtain 4 irred. reprs of A_7 of dimension 1, 15, 21, 35 coming from the 8 irred reprs of S_7 of dim. 1, 1, 15, 15, 21, 21, 35, 35.

Next we look at the restriction of the irred. reprs of S_7 of dimension 6, 6, 14, 14, 14, 14, 20.

If ρ is the 20-dimensional representation, then $\chi_\rho = \chi_\rho \cdot \lambda$ (or we would have another irred repr. of S_7 of dimension 20). So ρ_{A_7} must be reducible.

we obtain two irreducible (inequivalent) representations of A_7 of dimension 10.

- So far we got 6 irred. inequivalent reprs of A_7, but A_7 has 9 conjugacy classes, so there are 3 missing representations...

They come from the 6 irred reprs of S_7 of dimension

$$6,\ 6,\ 14,\ 14,\ 14,\ 14.$$

The corresponding characters must be of the form:

$$\chi_{\rho_1},\ \lambda\cdot\chi_{\rho_1},\ \chi_{\rho_2},\ \lambda\cdot\chi_{\rho_2},\ \chi_{\rho_3},\ \lambda\cdot\chi_{\rho_3}$$

for some irred. representations ρ_1, ρ_2, ρ_3 (of S_7) of dimension 6, 14 and 14. The restriction of ρ_1, ρ_2, ρ_3 to A_7 are irreducible. So we obtain 3 irreducible inequivalent reprs. of A_7 of dimension: 6, 14, 14.

The complete list of degrees of the irreducible characters of A_7 is:

$$1,\ 6,\ 10,\ 10,\ 14,\ 14,\ 15,\ 21,\ 35.$$
