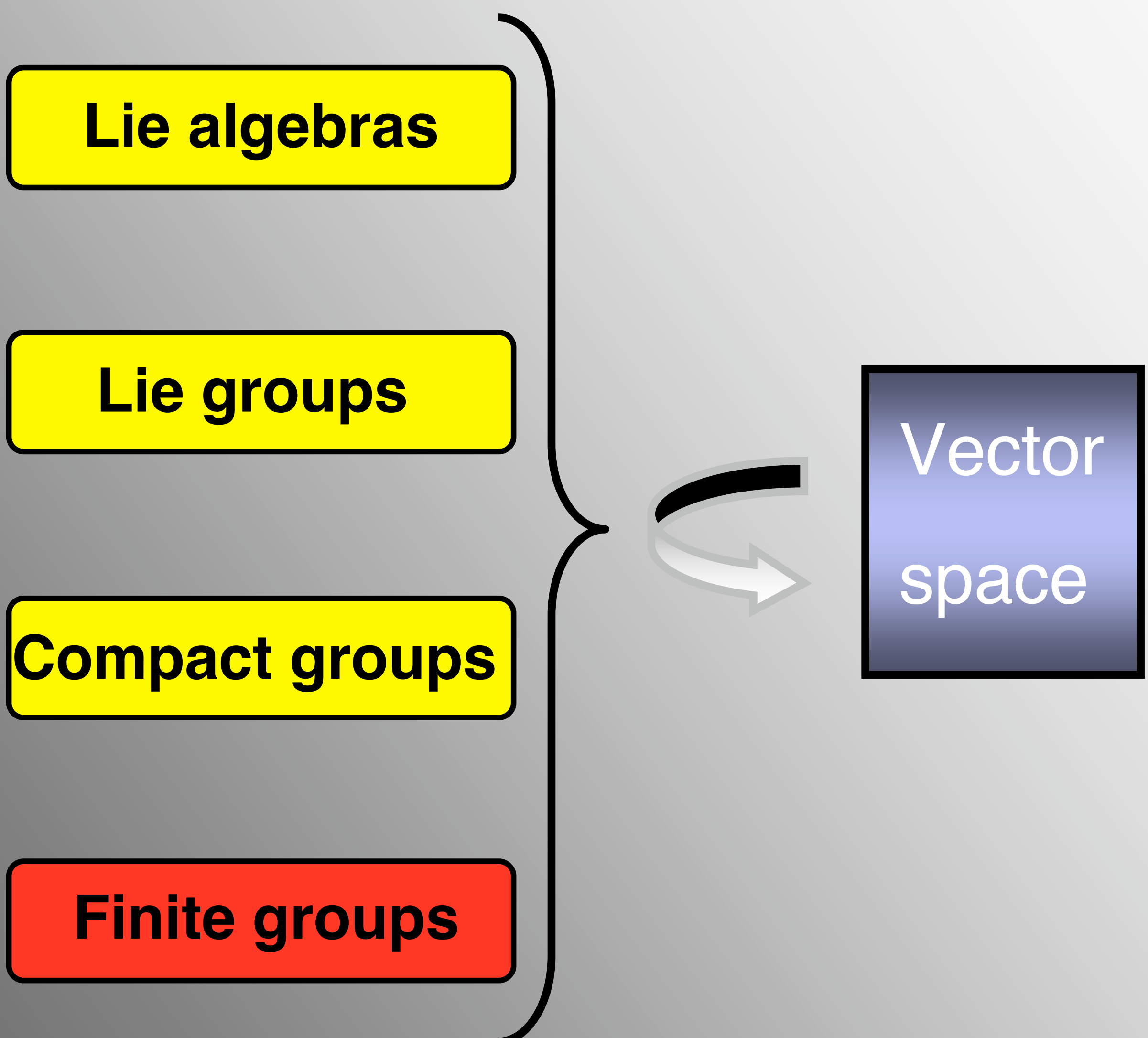
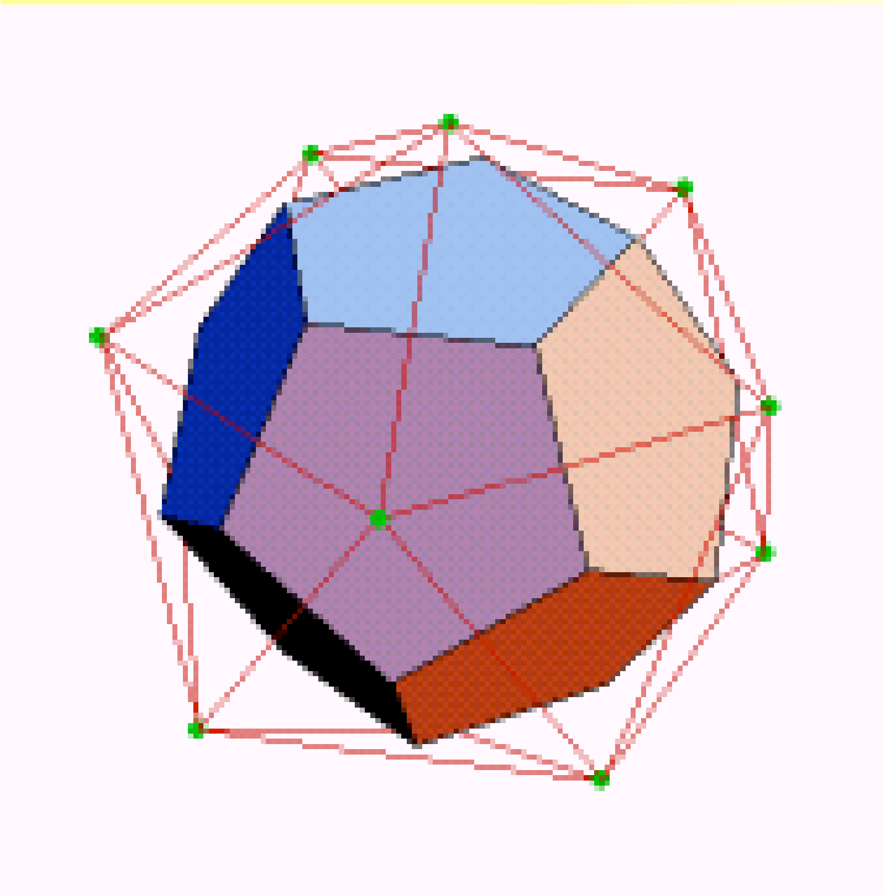
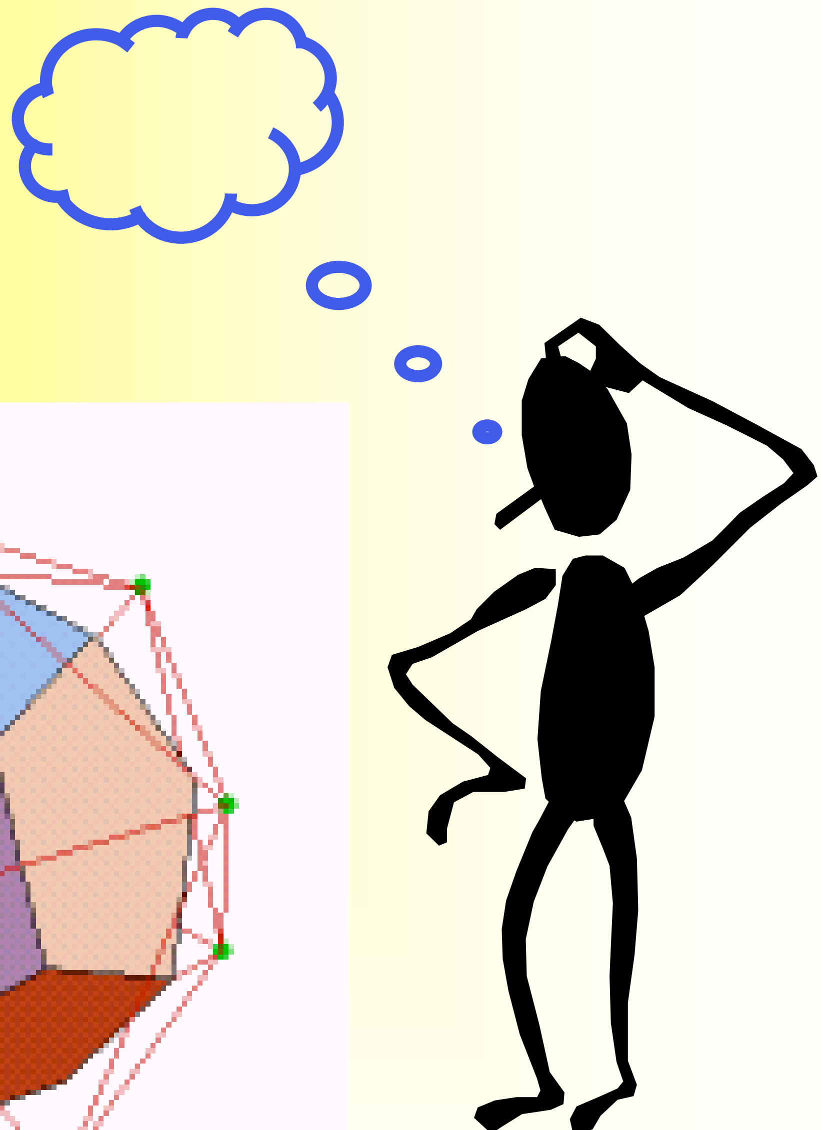
Representation Theory



Finite subgroups of SO(3)





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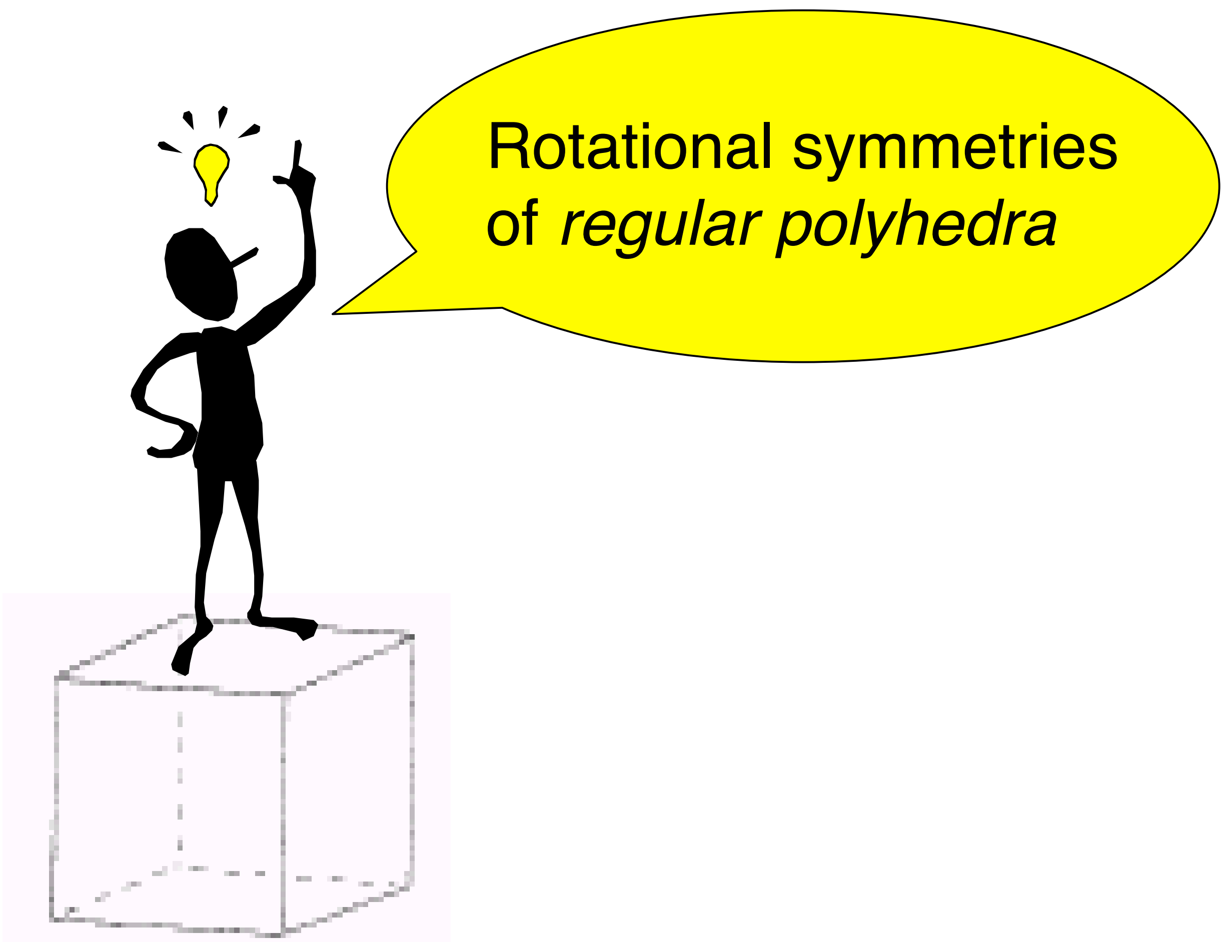
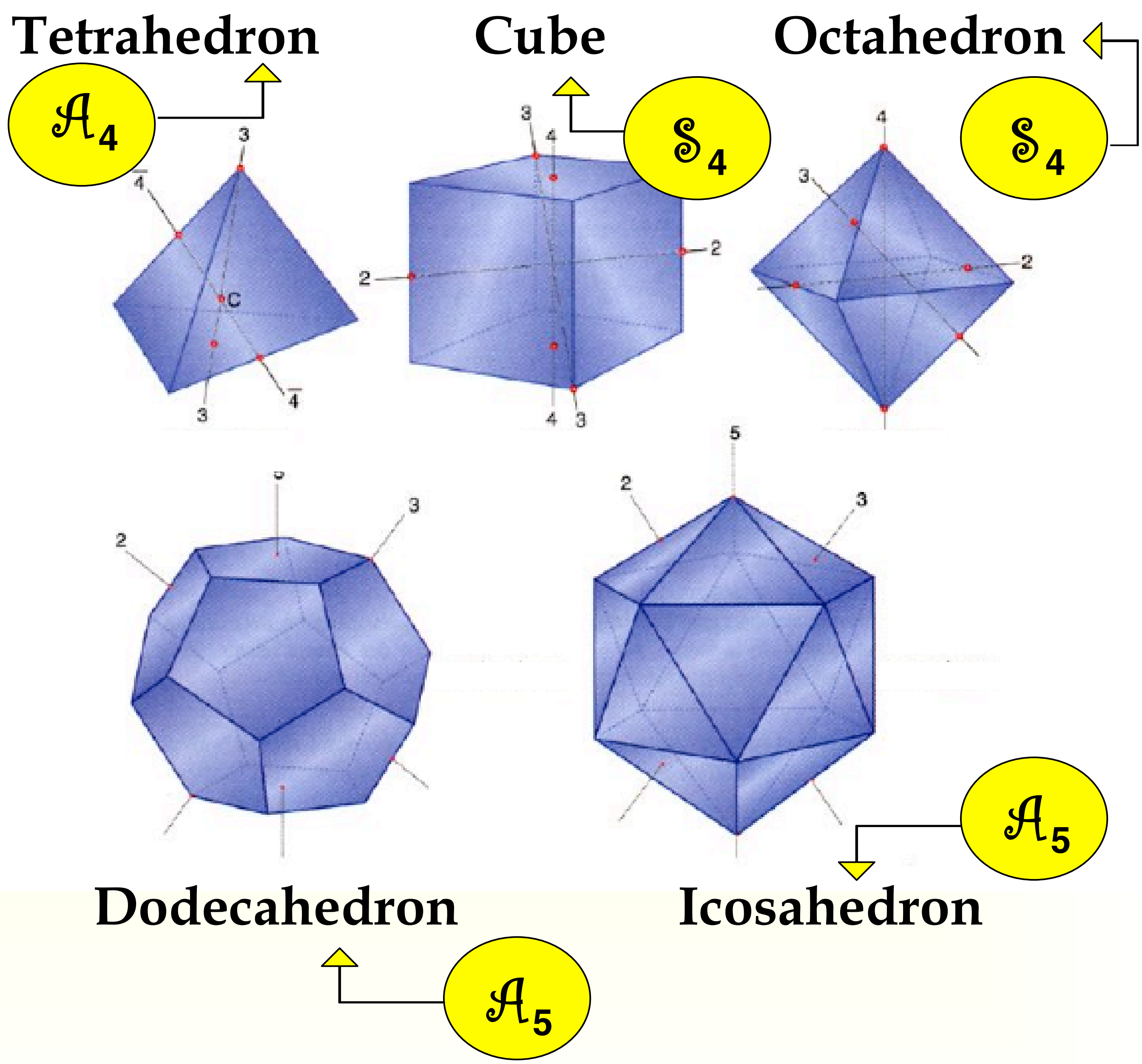
SO(3)= Rotations in 3D

How do we find **finite** subgroups?

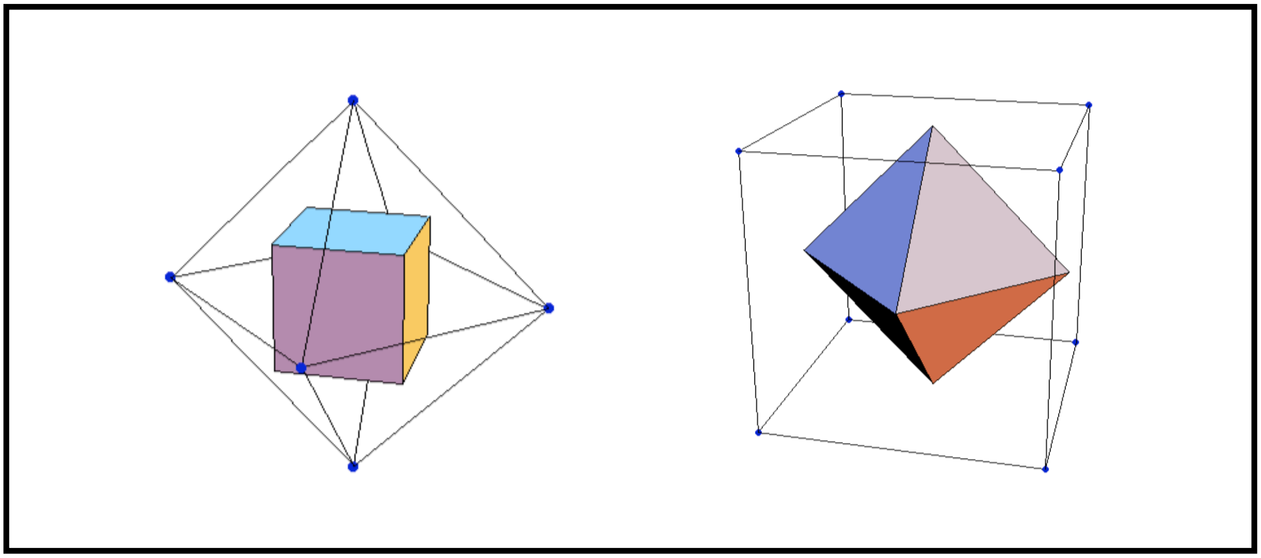


Regular Polyhedra

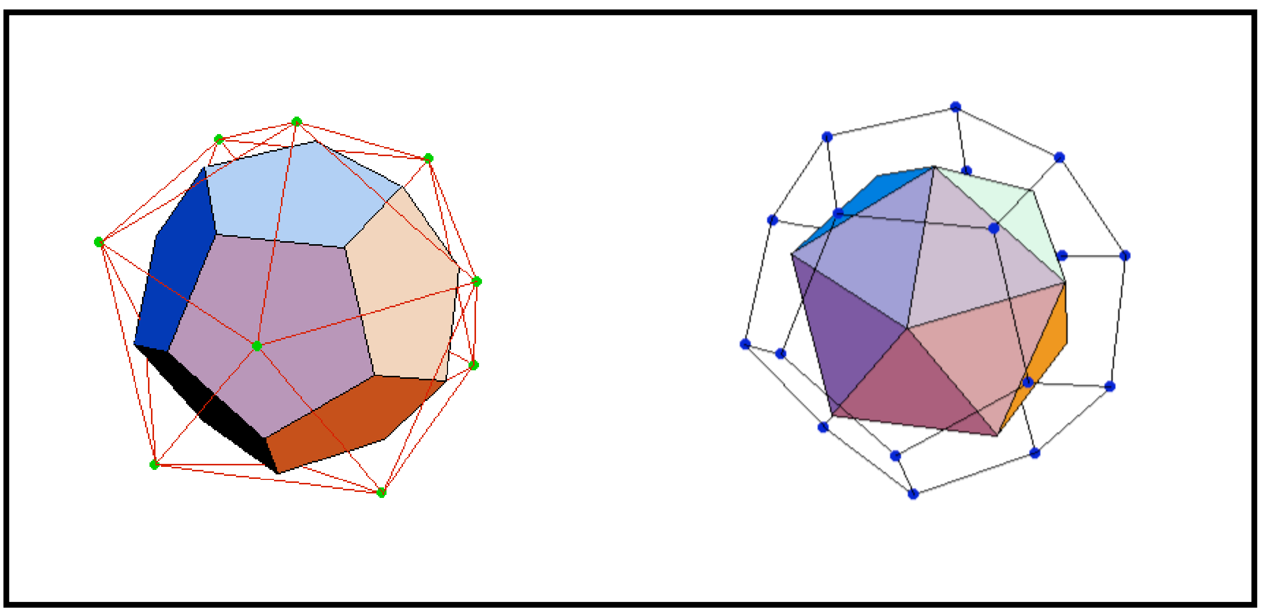


We obtain 3 distinct groups of rotations!!!

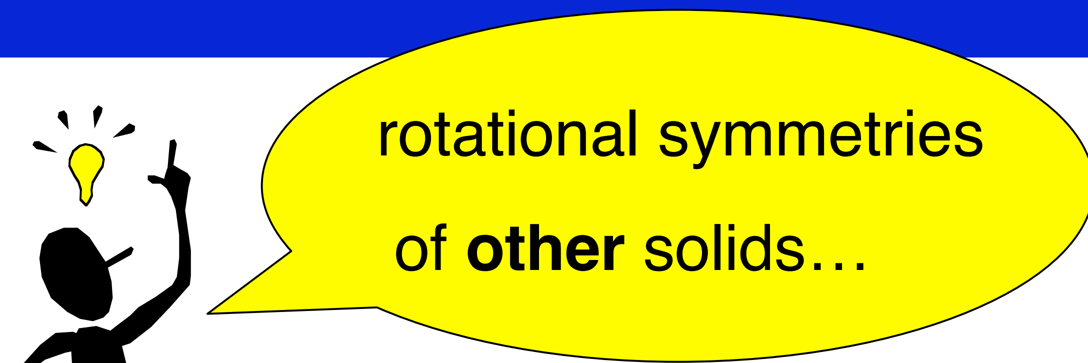
Why do we **only** get **3** groups?

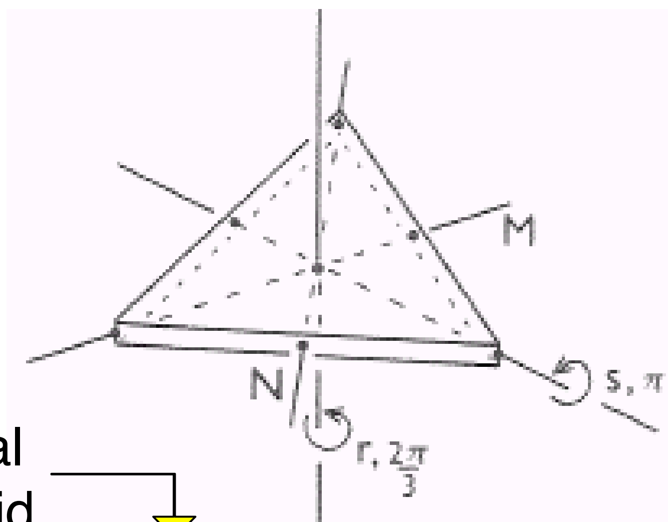


Duality of regular polyhedra: #V ⟺ #F



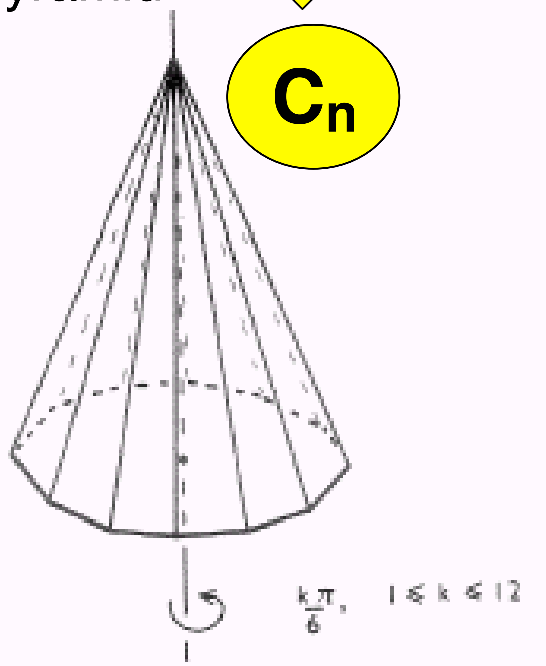
Are there other finite subgroups?

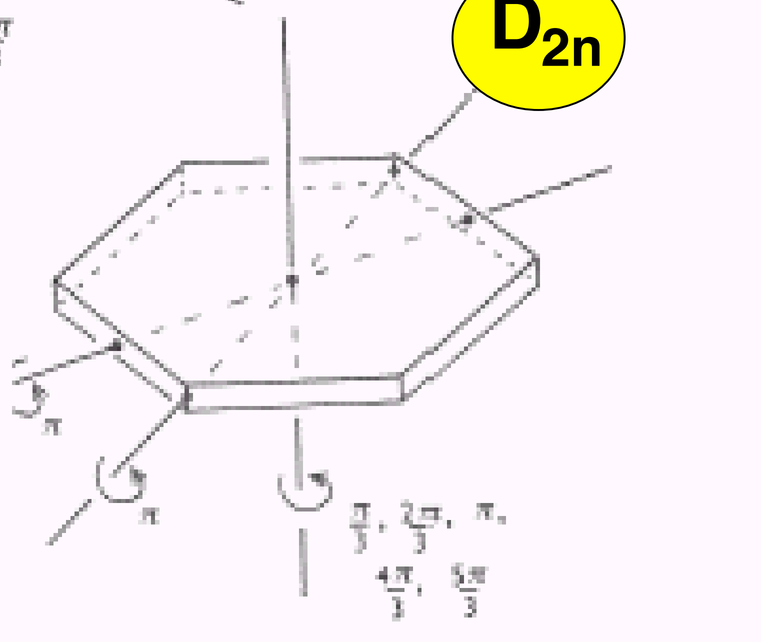




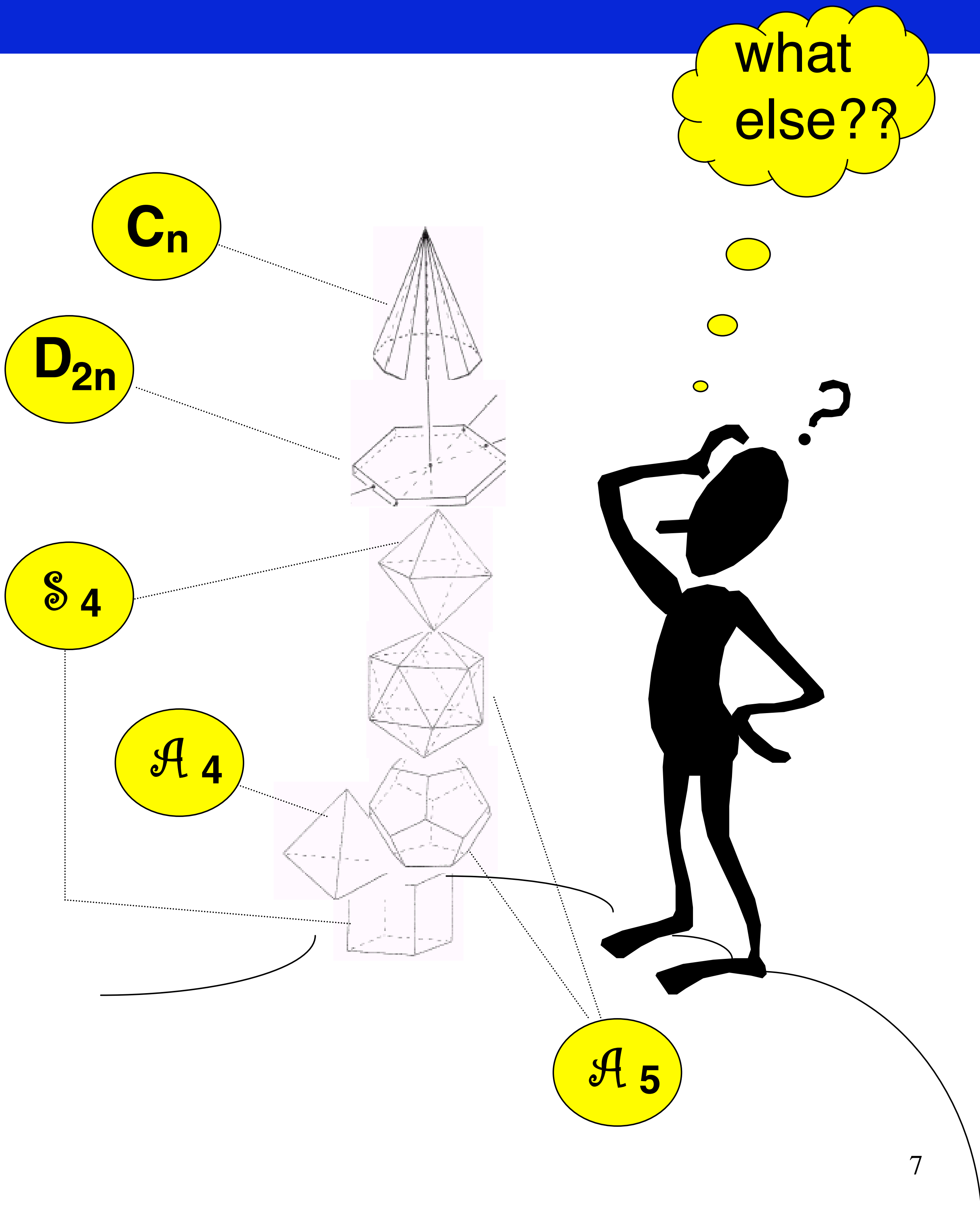
n-gonal plate → D_{2n}

n-gonal pyramid → C_n





Finite subgroups of SO(3)



A beautiful theorem!!!!!

Theorem: A finite subgroup of SO(3) is either cyclic, or dihedral, or it is the group of rotations of a platonic solid.

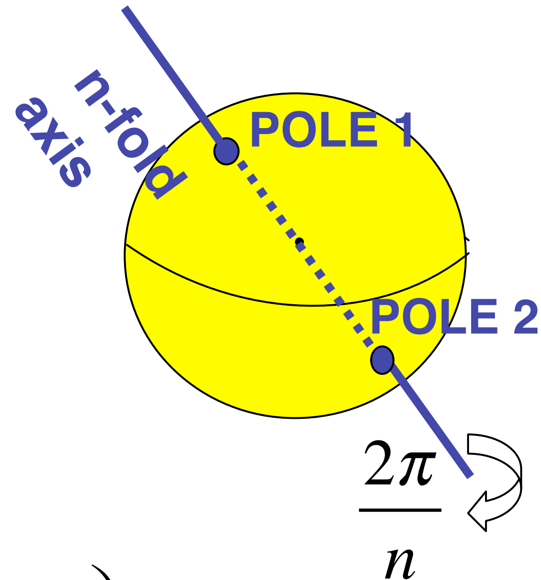
Finite subgroups of SO(3)

G < SO(3) finite

⇓

G acts on "poles"

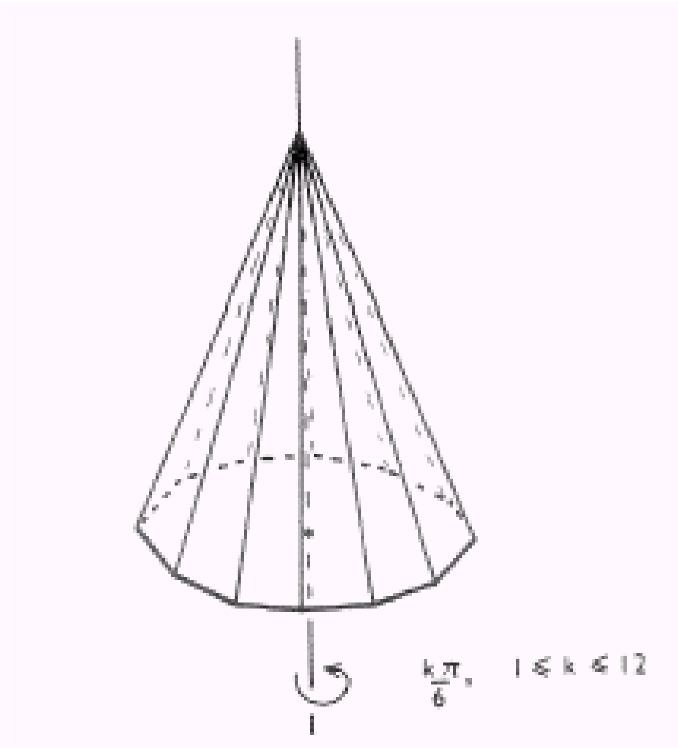
⇓



$$2\left(1-\frac{1}{|G|}\right)=\sum_{i=1}^{\#O}\left(1-\frac{1}{\left|st\left(\mathcal{O}_i\right)\right|}\right)$$

# P	*# O*	*order stabilizers*			*IGI*	*name*	*realization*
2	2	n	n		n	C_n	pyramid
2n+2	3	n	2	2	n	D_n	plate
26	3	4	3	2	24	S_4	cube
14	3	2	3	3	12	A_4	tetrah.
62	3	2	3	5	60	A_5	dodecah.

Symmetries of the pyramid



n = # sides in base

⇓

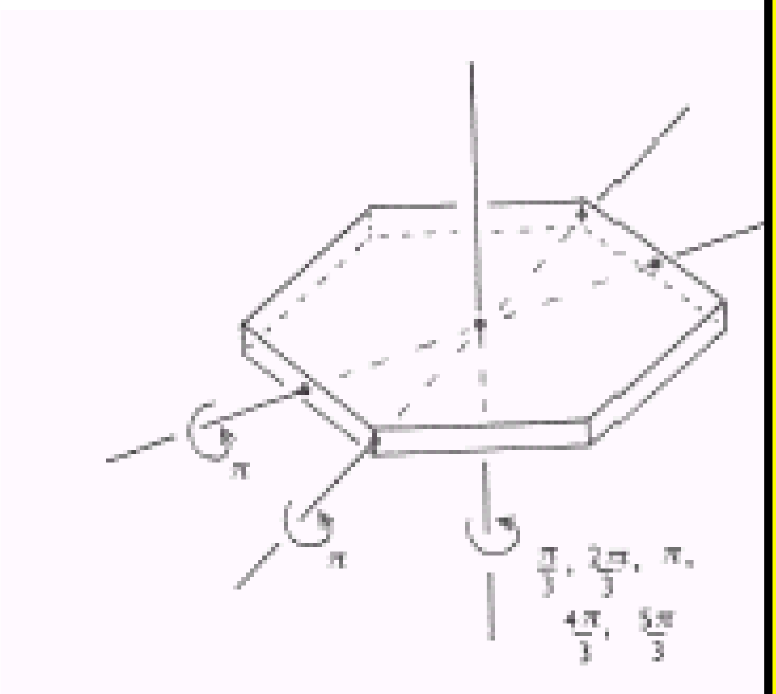
1 n-fold axis

⇓

G = cyclic group C_n

- # poles = 2
- # orbits = 2
- order of each stabilizer = n
- order of the group = n

Symmetries of the plate



n = # sides

⇓

n 2-fold axes

+

1 n-fold axis

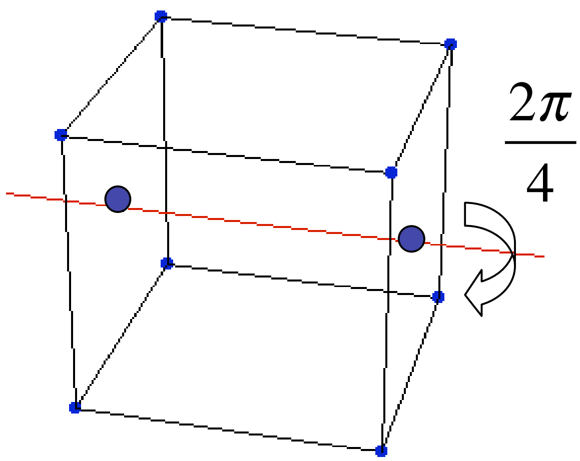
⇓

G = dihedral D_{2n}

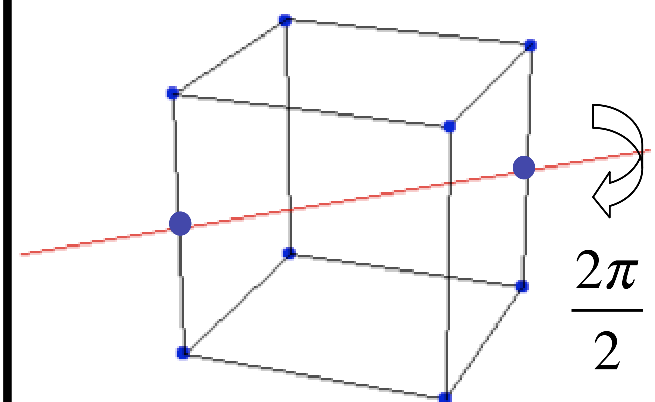
- # poles = 2n+2
- # orbits = 3
- order of stabilizers= 2, 2, n
- order of the group = 2n

Symmetries of the cube

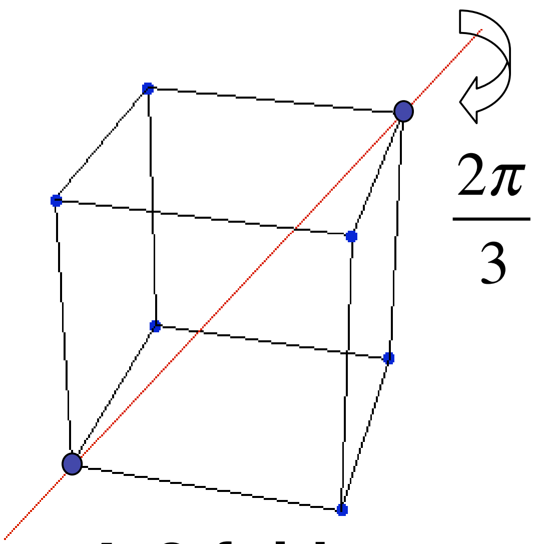
(8 vertices, 12 edges, 6 faces)



3 4-fold axes



4 2-fold axes



4 3-fold axes

$|G| = 1 + 3\times3 + 4 + 4\times2 = 24 = 4!$

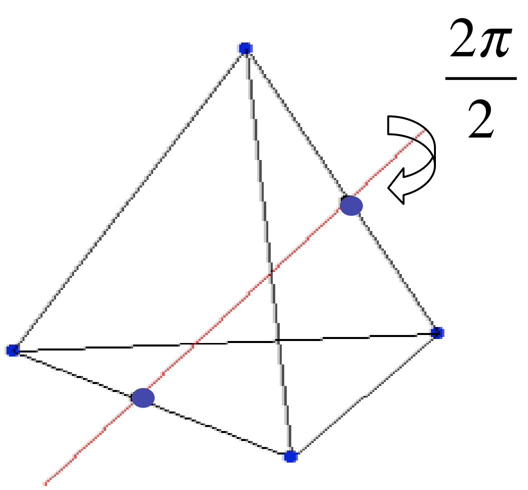
G permutes the 4 diag.s

$\Rightarrow G < \mathcal{S}_4$

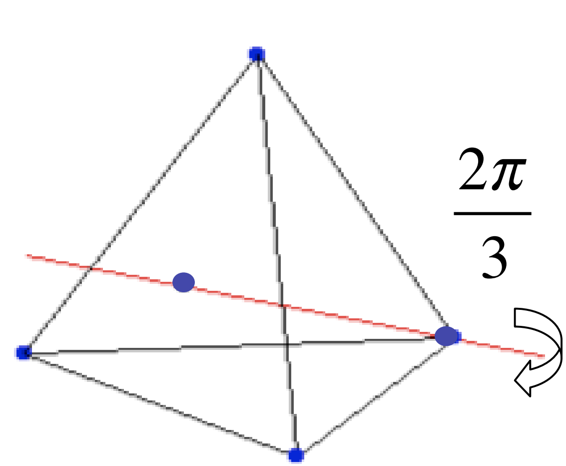
$|G| = 4! \Rightarrow$ **$G = \mathcal{S}_4$**

- # poles = 26; # orbits = 3
- order of stabilizers= 4, 2, 3
- order of the group = 24

Symmetries of the tetrahedron
(4 vertices, 6 edges, 4 faces)



3 2-fold axes



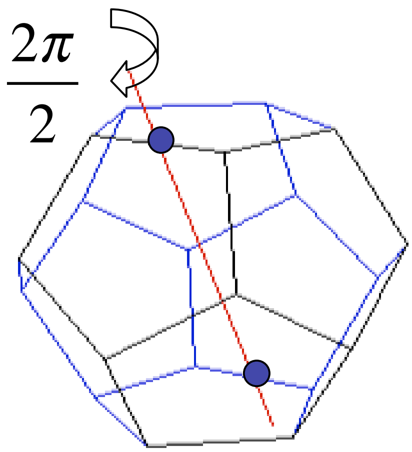
4 3-fold axes

- G permutes the 4 vertices ⇨ $G < \mathbb{S}_4$
- G generated by (...) and (..)(..) ⇨ $G < \mathcal{A}_4$
- IGI = 4!/2 ⇨ $G = \mathcal{A}_4$

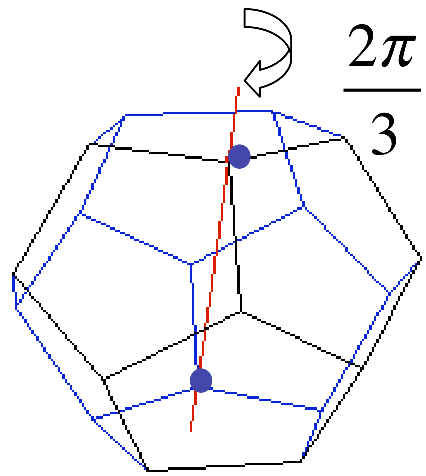
- # poles = 14 (6 centers of an edge, 4 vertices, 4 centers of a face)
- # orbits = 3
- order of stabilizers= 2, 3, 3
- order of the group = 1+ 3 x 1 + 4 x 2 = 12

Symmetries of the dodecahedron

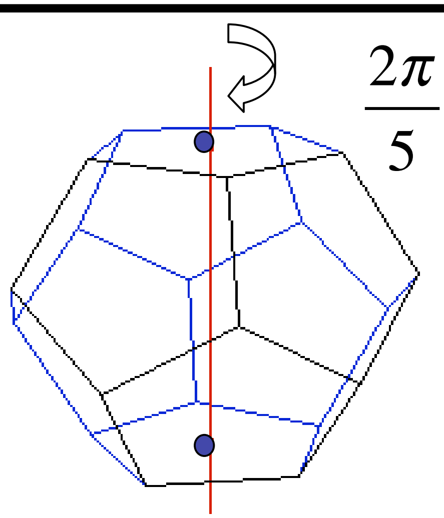
(20 vertices, 30 edges, 12 faces)



15 2-fold axes



10 3-fold axes



6 5-fold axes

$|G| = 1+15+10\times2 +$

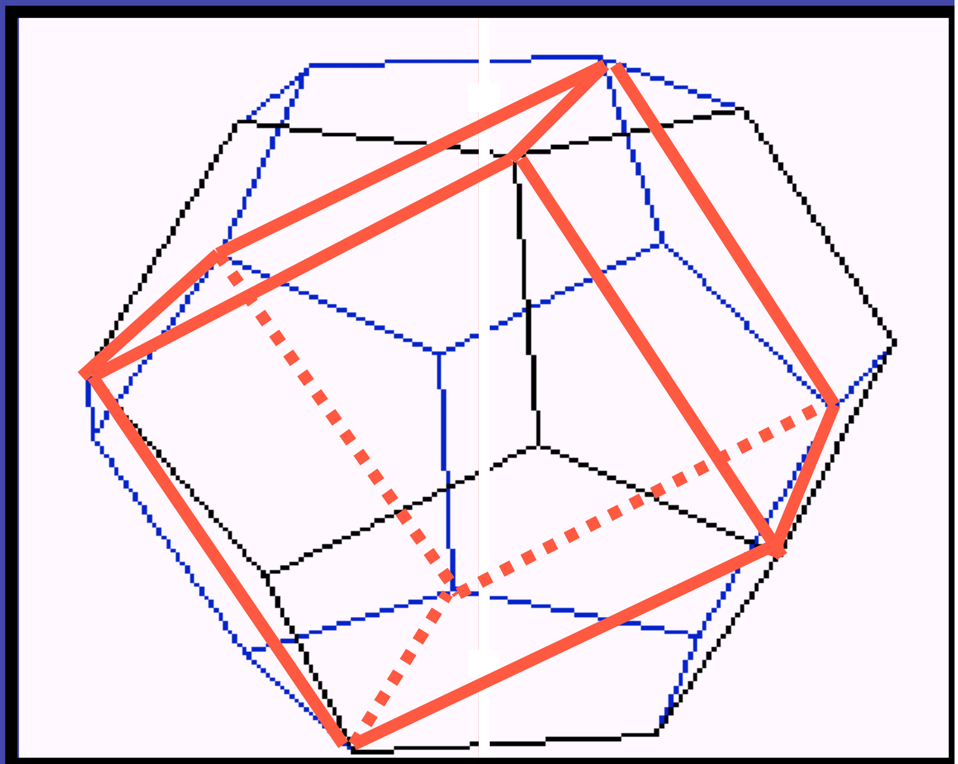
$+ 6\times4 = 60 = 5!/2$

$G = \mathcal{A}_5$

- # poles = 62; # orbits = 3
- order of stabilizers= 2, 3, 5
- order of the group = 60

Symmetries of the dodecahedron

There are exactly 5 cubes s.t. each edge of the cube is a diagonal of exactly 1 pentagon.



G permutes the 5 cubes ⟹	$G < \mathcal{S}_5$
G contains all (...) ⟹	$G > \mathcal{A}_5$
$\|G\| = 60 = 5!/2$ ⟹	$G = \mathcal{A}_5$