ALGEBRA QUALIFYING EXAM

June 18, 2012

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let $\mathbb{Z}$ denote the integers. Let $\mathbb{Q}$ denote the rational numbers. Let $\mathbb{R}$ denote the real numbers. Let $\mathbb{C}$ denote the complex numbers. Let S_n denote the symmetric group on n letters.

1. Recall that the *exponent* of a group is the smallest positive integer n such that $g^n = 1$ for every $g \in G$.
   (a) Compute the exponent of the multiplicative group $(\mathbb{Z}/77\mathbb{Z})^\times$.
   (b) Compute the exponent of S_5.
2. Fix a prime p. For all positive integers m, n, let $f(m,n)$ be the number of nonzero ring homomorphisms from $\mathbb{F}_{p^m}$ to $\mathbb{F}_{p^n}$.
   (a) What is $f(m,6)$?
   (b) What is $f(6,n)$?
3. Show that a group with exactly 3 elements of order 2 is not simple.
4. List all the ideals in the ring $\mathbb{Q}[x]/(x^3 - x^2 - x + 1)$.
5. Let $L = \mathbb{Q}(\sqrt[6]{-3})$. Show that $L/\mathbb{Q}$ is Galois and $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$.
6. Give the character table (over $\mathbb{C}$) of the quaternion group Q_8. Justify your answer.
7. Suppose R is a commutative ring and $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules. Show that if A and C are finitely generated, then so is B.
8. Which of the following matrices are similar over $\mathbb{Q}$? Which are similar over $\mathbb{C}$?
$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$
9. Short answer. For each of the following give the answer and a *brief* explanation.
   (a) True or False: If G is a group, and $g, h \in G$ both have finite order, then gh has finite order.
   (b) True or False: The group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ has exactly 2 subgroups of index 2.
   (c) True or False: The group S_4 is solvable.
   (d) True or False: If R is a commutative ring and M, N are nonzero R-modules, then $M \otimes_R N$ is nonzero.
   (e) Are the fields $\mathbb{Q}(\sqrt{7})$ and $\mathbb{Q}(\sqrt{11})$ isomorphic (as fields)? Explain.
10. For each of the following, either give an example or explain *briefly* why no such example exists:
    (a) a group G in which the set of squares in G is not a subgroup.
    (b) an element in $\mathrm{GL}_n(\mathbb{Q})$ of order 15.
    (c) an extension $F/\mathbb{R}$ of degree 4.
    (d) a commutative ring with unity which is not a field and which has exactly one prime ideal.
    (e) a non-principal ideal in $\mathbb{Z}[\sqrt{-13}]$.