ALGEBRA QUALIFYING EXAMINATION

June, 2001

There are 11 questions, but you are not expected to answer all of them. Do as many problems as you can. We prefer complete solutions of a few problems to many partial solutions. 60 total points is sufficient to pass at the Master's level. 75 total points is sufficient to pass at the Ph.D. level The number of points for each section of a problem (or for the whole problem, if it is not subdivided into sections) is given in parenthesis. Even if the problem is not subdivided into sections, partial credit may be given. You have 150 minutes. GOOD LUCK!

Notation: If R is ring, R^* denotes the multiplictive group of units of R. $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}$ denote the complex, real, rational and integer numbers rpectively. $\mathbf{F}_q$ denotes the field with q elements.

The symmetric group S_n

1. Let G be a subgroup of S_n of order $(n-1)!$. S_n acts by left multiplication on the coset space S_n/G.

   (3) a. Explain how this action corresponds to a homomorphism $h : S_n \to S_n$.

   (3) b. Let $K = \text{Ker}(h)$. Show that $K \subset G$.

   (3) c. Assuming $K = \{1\}$, explain how h identifies G with S_{n-1}.

   (3) d. Prove that $K = \{1\}$. You may assume that $n \geq 5$, so that A_n (the alternating group) is simple. Then use the fact that K is normal.

Semi-Direct Products

2. Let $p_1 < p_2$ be primes, and let G be a group of order p_1p_2.

   (4) a. Show that G is a semi direct product of a subgroup H of order p_1 and a normal subgroup N of order p_2. (Use the Sylow theorems.)

   (4) b. Show that if p_1 does not divide $(p_2 - 1)$, then G is a direct product, and is therefore abelian. (HINT: The semi-direct product structure is determined by a homomorphism $\phi : H \to \text{Aut}(N)$; consider the order of $\text{Aut}(N)$)

   (4) c. Show that if p_1 divides $(p_2 - 1)$ then it is possible that G is nonabelian. HINT: show that there is a canonical nonabelian semi-direct product structure on $\mathbf{Z}/p_2 \times (\mathbf{Z}/p_2)^*$ in which $\mathbf{Z}/p_2 \times \mathbf{Z}/p_1$ may be viewed as a subroup (because $\mathbf{Z}/p_1$ may be viewed as a subgroup of $(\mathbf{Z}/p_2)^*$ in this case.)

Group Representations

3. Let G be a finite group. Let ρ be a representation of G on $\mathbf{C}^d$.

   4) a. Show that ρ is conjugate to a unitary representation (i.e., there exists a basis for $\mathbf{C}^d$ in terms of which the matrix which represents $\rho(g)$ is unitary for all g in G).

   4) b. Show that if λ is an eigenvvalue of a unitary operator, that $|\lambda| = 1$.

   (4) c. Prove: For G, ρ, d be as above, let χ be the character of ρ. Show that for all g in G, $|\chi(g)| \leq d$, and if $|\chi(g)| = d$ then $\chi(g) = \zeta d$ where ζ is a root of unity.

4. Let G be a finite group of order n. Let r denote the number of conjugacy classes in G, let s denote the number of isomorphism classes of irreducible finite dimensional complex representations of G, and let $d_1, ..., d_s$ be the dimensions of the irreducible representations in those distinct isomorphism classes.

(4) a. State the numerical relationships among the numbers $n, r, s, d_1, ..., d_s$ which hold in general

(4) b. For the case $G = S_3$ (the symmetric group) give the character table

(i.e. give the character for each isomorphism class of irreducible representations).

(4) c. Using a., show that if G is abelian then every irreducible representation is one dimensional.

Finite Fields

5. Let $\mathbf{F}_q$ be a field of characteristic p with q elements. Let $\alpha = [\mathbf{F}_q:\mathbf{F}_p]$.

(2) a. Express q in terms of α and p; justify.

(3) b. Show that every extension of $\mathbf{F}_p$ is separable.

(3) c. Show that $\mathbf{F}_q$ is a Galois extension of $\mathbf{F}_p$: Find a polynomial over $\mathbf{F}_p$ satisfied by every element of $\mathbf{F}_q$ (justify your answer). Conclude that all fields with q elements are isomorphic.

(4) d. Find an automorphism ϕ of $\mathbf{F}_q$ over $\mathbf{F}_p$ with exponent α. Conclude that $G(\mathbf{F}_q/\mathbf{F}_p)$ is cycic of degree α.

The Euler Function

6. Let E denote the function which assigns to an integer n the number of positive integers $< n$ which are relatively prime to n.

(4) a. Calulate $E(p^r)$ where p is prime.

(3) b. If R_1 and R_2 are rings, show that $(R_1 \times R_2)^* = R_1^* \times R_2^*$.

(5) c. If the prime factorizaion of n is $p_1^{r_1} \cdots p_k^{r_k}$, give s formula for $E(n)$ in terms of $p_1, ..p_k, r_1, .., r_k$. Justify your answer.

Galois Groups over Q

7. (12) Find the Galois group (of the splitting field) of the polynomial $X^5 - 2$ over $\mathbf{Q}$. Justify.

Irreducibility of a Poynomial over Q

8. (12) Consider the polynomial $X^5 - 9X + 2$ over $\mathbf{Q}$. Show it is irreducible over $\mathbf{Q}$. (HINT: Check irreducibility modulo a suitable prime. Depending on what prime you chose, you may need to also check that there are no rational roots).