Algebra Qualifying Exam

August 1997

Do six of the following ten problems.

In this exam $\mathbb{Z}$ denotes the ring of integers of the field $\mathbb{Q}$ of rational numbers, $\mathbb{R}$ denotes the field of real numbers, and $\mathbb{F}_n$ a finite field.

1. Let G be a finite group with the property that for any two subgroups H and K, either $H \subseteq K$ or $K \subseteq H$. Prove that G is a cyclic group of order $|G| = p^n$, p prime.

2. Prove that no group G of order $5^n.6$, $n \geq 2$, is simple.

3. Let H be the subgroup of the general linear group $GL(2, F)$, of 2×2 invertible matrices over a field F, that stabilizes the set $\{\binom{x}{0} \mid x \in F\}$ under left multiplication. Prove that H is a solvable group.

4. Find the greatest common divisor $d(X)$ of the polynomials

$$f(X) = X^4 - X^2 + 2X - 1 \text{ and } g(X) = X^4 + 2X^3 + X^2 - 1$$

over the field of real numbers $\mathbb{R}$. Find polynomials $u(X)$ and $v(X)$ such that

$$d(X) = u(X)f(X) + v(X)g(X).$$

5. Determine the structure (as a direct product of cyclic groups) of the group of units in the ring $\mathbb{F}_5[u]$ where $u^3 = 1$ ($\mathbb{F}_5$ denotes the finite field with 5 elements).

6. Let D be the ring of Gaussian integers $\mathbb{Z}[i]$, and $M = D^3$ the free D-module of rank 3. Take K to be the submodule generated by $(1, 2, 1), (0, 0, 5)$ and $(1, -i, 6)$. Prove that M/K is finite, and determine its order.

7. Let E be the splitting field of $X^{35} - 1$ over the finite field $\mathbb{F}_8$ with 8 elements. Determine the cardinality $|E|$ of E. How many subfields does E have?

8. Determine the degree $[E : \mathbb{Q}]$ of the splitting field E of $X^{10} - 5$ over the rational field $\mathbb{Q}$.

9. Let F be a field and let $f(X) \in F[X]$ be a separable irreducible polynomial of degree 4. Determine, as explicitly as possible, the Galois group G, of the splitting field of $f(X)$ over F, when G has order 8.

10. Show that the splitting field E of the polynomial

$$f(X) = X^3 + X^2 - 2X - 1$$

over the rational field $\mathbb{Q}$ is obtained by adjoining a single root of $f(X)$. Find the Galois group Gal $(E/\mathbb{Q})$.

HINT: Show first that $f(X)$ divides $f(X^2 - 2)$.