Qualifying Exam in Algebra

May 1997

Do six of the following ten problems.

In this exam $\mathbb{Z}$ denotes the integers of the field $\mathbb{Q}$ of rational numbers.

You may cite without proof only results proved in Math 535-536, or proved in Chapters 1-4 of Jacobson's "Basic Algebra".

1. If N is a finite normal subgroup of a group G, and if G/N contains an element of order n, prove that G contains an element of order n.

2. Assume that p is an odd prime and G is a finite simple group with exactly $2p+1$ Sylow p-groups. Prove that the Sylow p-groups of G are abelian.

3. Suppose that $G' \subseteq Z(G)$ for some group G where G' is the commutator subgroup and $Z(G)$ the center. Show that, for all $g \in G$, the function

$$f_g(x) = xgx^{-1}g^{-1}$$

is a homomorphism.

4. Let $R = \mathbb{Z}[X, Y]$ be the ring of polynomials in two indeterminants X, Y over the rational integers. For each of the following five ideals of R with given generators, determine whether or not it is prime, and whether or not is is maximal.

(a) (X, Y)	(b) $(X, 3Y)$	(c) (X^2+1, Y)
(d) $(3, X^2+1, Y)$	(e) $(5, X^2+1, Y)$.	

5. Let R be a commutative ring with identity, and let $f(X), g(X)$ be elements of the polynomial ring $R[X]$. Assume the ideals generated by the coefficients of $f(X)$, and of $g(X)$, are both R. Prove that the ideal generated by the coefficients of the product $f(X)g(X)$ is also R.

6. The finite field $\mathbb{F}_{64}$ with 64 elements has how many elements of multiplicative order 9? Support your answer.

7. Let M be the $\mathbb{Z}$-module generated by a, b, c with the relations

$$4a + 3b + 3c = 2a - b + 3c = 0.$$

Express M as a direct sum of cyclic modules. What are the orders of these modules?

8. Find the minimum polynomial of a primitive complex 20-th root of unity over the rational field $\mathbb{Q}$.

9. Let E be the splitting field of $X^{42} - 1$ over the rational field $\mathbb{Q}$. Determine the number of subfields of E.

10. Let F be a field of characteristic zero not containing a primitive n-th root of unity. Assume $f(X) = X^n - a$, $a \in F$, is irreducible. Show that the Galois group of the splitting field of $f(X)$ over F is isomorphic to a group of linear transformations of the form

$$z \mapsto bz + c$$

where $b, c \in \mathbb{Z}/n\,\mathbb{Z}$.