1. Assume $f \in L^1(\mathbb{R}, m)$ where m is the Lebesgue measure. Let $E_n = \{x : f(x) > n^2\}$. Prove or disprove

   a) $m(E_n) \to 0$ as $n \to \infty$

   b) $\int_{E_n \cup [n,2n]} f dm \to 0$ as $n \to \infty$

   c) there is a set B of Lebesgue measure 0 such that $f(x) \to 0$ as $x \to \infty$ for $x \notin B$.

2. Let f be a Lebesgue measurable extended real valued function on $[0,1]$. Let $p \in [1, \infty)$ and $r > 0$, and suppose we have $\int_0^1 x^{-r} |f(x)|^p \, dx < \infty$. Prove that

$$\lim_{t \downarrow 0} t^{-\left(1+\frac{r-1}{p}\right)} \int_0^t f(x) \, dx = 0.$$

3. Consider the Lebesgue measure in $\mathbb{R}^2$. Suppose $E \subset [0,1]^2$ is measurable. Let

$$E_x = \{y \in [0,1]; \; (x,y) \in E\} \quad \text{and} \quad E_y = \{x \in [0,1]; \; (x,y) \in E\}.$$

   Show that if $\mu_L(E_x) = 0$ for μ_L-a.e. $x \in [0, \frac{1}{2}]$, then

$$\mu_L \left(\{y \in [0,1]; \; \mu_L(E_y) = 1\}\right) \leq \tfrac{1}{2}.$$

4. Show that for a.e. $x \in [0,1]$ and for every $\gamma > 2$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{n^\gamma} \frac{1}{\sqrt{|x - \frac{k}{2^n}|}} < +\infty.$$

5. Let μ be a measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure m and with Radon-Nikodym derivative equal to e^{-x}. Assume that f_n are μ-measurable functions with $\int_{\mathbb{R}} |f_n| d\mu \leq \frac{1}{n^\gamma}$ for all $n \in \mathbb{N}$. Prove or disprove that for Lebesgue a.e. $x \in [0,1]$,

$$\lim_{n \to +\infty} f_n(x) = 0.$$

   if

   a) $\gamma = 1$,

   b) $\gamma = 2$

6. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f_n \in L^1(X, \mathcal{A}, \mu)$. If $f_n \to f$ μ-a.e., show that there exist sets $H, E_k \in \mathcal{A}$ such that $X = H \cup (\cup_{k=1}^{\infty} E_k)$, $\mu(H) = 0$ and $f_n \to f$ uniformly on each E_k.