Print Your Name: ______________ ______________
last first

Print Your I.D. Number: __________ ______________

Qualifying Examination, June 18, 2014
10:00 am–12:30pm, Room RH 114

Choose any 8 problems from 9

Table of your scores

Problem 1 ________/ 10

Problem 2 ________/ 10

Problem 3 ________/ 10

Problem 4 ________/ 10

Problem 5 ________/ 10

Problem 6 ________/ 10

Problem 7 ________/ 10

Problem 8 ________/ 10

Problem 9 ________/ 10

Total ________/ 80

Notation: $D(z_0, r)$ denotes the open disc in $\mathbf{C}$ centered at z_0 and radius r.

1. Complete the following two problems.

(a) Describe all entire holomorphic functions f with $|f(z)| \leq |z|$ for all $z \in \mathbf{C}$;

(b) Describe all entire holomorphic functions f with $\lim_{z\to\infty} \frac{f(z)}{z} = 0$.

2. Complete the following two problems.

(a) Evaluate $\int_{|z|=1} \exp(\frac{1}{z^2})dz$. (Here, $\exp(z) =: e^z$)

(b) Evaluate $\int_0^\infty \frac{x^2}{1+x^4}dx$

3. Let $f : D(0,1) \to D(0,1)$ be holomorphic with $f(0) = \frac{1}{3}$.

(a) Give a sharp upper bound estimate for $|f'(0)|$

(b) Give an example of f such that $|f'(0)|$ achieves the upper bound you obtained in Part (a)

4. Prove that there is an N such that if $n \geq N$ then

$$\sum_{k=0}^{n} (k+1) z^k \neq 0, \quad z \in D(0, 3/4)$$

5. Let $f_1, \cdots, f_n$ be holomorphic in a domain D in $\mathbf{C}$ and $p \in (0, \infty)$. Prove

(a) $\sum_{j=1}^{n} |f_j(z)|^p$ is subharmonic in D

(b) if there is a $z_0 \in D$ such that $\sum_{j=1}^{n} |f_j(z_0)|^p \geq \sum_{j=1}^{n} |f_j(z)|^p$ for all $z \in D$, then f_j is constant for $j = 1, 2, \cdots, n$.

6. Let $D = \{z \in \mathbf{C} : 1 < |z+1| \text{ and } \ |z+2| < 2\}$. Construct a conformal holomorphic map which maps D onto the unit disc $D(0,1)$

7. Let D be a simply connected domain in $\mathbf{C}$ and $z_0 \in D$. If $\phi_1, \phi_2 \in \mathrm{Aut}(D)$ such that

$$\phi_1(z_0) = \phi_2(z_0) \quad \text{and} \quad \phi_1'(z_0) = \phi_2'(z_0)$$

then $\phi_1 \equiv \phi_2$. (Hint: Try $D = D(0,1)$ and $D = \mathbf{C}$ first)

8. Let $f(z)$ be holomorphic in $D =: D(0,1) \setminus \{0\}$ such that

$$\int_D |f(z)| dA(z) < \infty$$

Prove that $z = 0$ is either removable or a simple pole .

9. Let $f_n : D(0,1) \to D(0,1) \setminus \{0\}$ be a sequence of holomorphic functions with $\sum_{n=1}^{\infty} |f_n(0)|^2 < \infty$. Prove that

$$\sum_{n=1}^{\infty} |f_n(z)|^3$$

converges uniformly on $\overline{D(0,1/5)}$.