Print Your Name: ____________ ____________
last first

Print Your I.D. Number: __________ ____________

Comprehensive Examination of Analysis

9:00Am–11:30AM, June 18, 2013

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1. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively by

$$a_1 > \frac{3}{2}, \qquad a_n = \sqrt{3a_{n-1} - 2}, \qquad n \geq 2,$$

converges and finds its limit.

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2. Show that the series

$$\sum_{n=1}^{\infty} \frac{x \sin(n^2 x)}{n^2}$$

converges pointwise to a continuous function on $\mathbb{R}$

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3. Prove the following integral test. Assume that f is a positive and decreasing function on the interval $(0, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the the sequence $\{I_n\}$ is bounded, where $I_n = \int_1^n f(x)\, \mathrm{dx}$.

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4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy

$$\lim_{|x| \to \infty} f(x) = 0.$$

Prove or disprove $f(x)$ is uniformly continuous on $\mathbb{R}$.

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5. Let f be an increasing function on $[0, 1]$. $D(f)$ denotes the set of all discontinuous points of f on $[0, 1]$. Prove $D(f)$ is at most countable.

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6. Evaluate the following integral

$$\int_{S^2} z^4 y^2 d\sigma$$

where $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and $d\sigma$ is the area element on S^2.

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7. Let f be a bounded function on $[a, b]$. Prove or disprove
(a) If $f(x)^2$ is integrable on $[a, b]$ then f is integrable.
(b) If $f(x)^3$ is integrable on $[a, b]$ then f is integrable.

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8. Let $f(x)$ be a twice differentiabale function on $[-1, 1]$ such that

$$f(0) = 0 \quad \text{and } f(1) = -f(-1)$$

Prove there is a $x_0 \in (-1, 1)$ such that $f''(x_0) = 0$.

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9. Let $C[0,1]$ be the metric space consisting of all continuous functions on $[0,1]$ with a metric d defined by

$$d(f,g) = \max\{|f(x) - g(x)| : x \in [0,1]\}.$$

Let h be a differentiable function on $\mathbb{R}$ with $|h'(x)| \leq 1/2$ for all $x \in \mathbb{R}$. A map $T : C[0,1] \to C[0,1]$ is defined by

$$T(f)(x) = (h \circ f)(x), \quad \text{for all } x \in [0,1] \quad \text{and } f \in C[0,1].$$

Prove that T has a unique fixed point in $C[0,1]$.