Print Your Name: ______________ ________________
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Comprehensive Examination of Analysis

9:00AM–11:30AM, June 16, 2015; Rowland Hall 114

Choose 8 from the 9 problems

You need to cross out the problem you don't want to be graded

Problem 1 ————/ 10

Problem 2 ————/ 10

Problem 3 ————/ 10

Problem 4 ————/ 10

Problem 5 ————/ 10

Problem 6 ————/ 10

Problem 7 ————/ 10

Problem 8 ————/ 10

Problem 9 ————/ 10

Total ————/ 80

Score:———/10 Your Name: ____________ ____________
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1. Prove that

$$\sum_{n\geq 2} \frac{1}{n(\log n)^2} < +\infty.$$

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2. Compute

$$\lim_{n\to+\infty} \int_0^1 \sin(nx)e^{-x^2}\,dx.$$

Justify your answer.

Score:———/10 Your Name: ____________ ____________
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3. Assume that $f \in C^1(\mathbb{R})$ and $\lim_{|x|\to+\infty} \frac{f(x)}{|x|} = +\infty$. Show that for any $p \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $f'(y) = p$ (i.e., $f' : \mathbb{R} \to \mathbb{R}$ is onto). Hint: Consider $g(x) = f(x) - px$ and $\lim_{x\to\pm\infty} g(x)$.

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4. (a) State the Stokes' Theorem

(b) Evaluate the following integral:

$$\int_{\partial D} \frac{x^3}{3} dy \wedge dz + \sin(yz) dy \wedge dz + x^{10} dx \wedge dz$$

where

$$D = \left\{ (x, y, z) : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} < 1 \right\}$$

Score:———/10 Your Name: ____________ ____________
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5. Prove that $\sin(\sqrt{x})$ is uniformly continuous on $[0, \infty)$.

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6. Let

$$f_n(x) =: \frac{nx}{1 + n^2 x^3}.$$

(a) Prove $f_n(x) \to 0$ as $n \to \infty$ pointwisely on $[0, \infty)$;
(b) Prove or disprove $f_n(x) \to 0$ as $n \to \infty$ uniformly on $[0, \infty)$.

Score:———/10 Your Name: ____________ ____________
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7. Let (X, d) be a metric space and $E \subset X$ is a compact set. Prove that E is closed.

Score:———/10 Your Name: ____________ ____________
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8. Let $f(x,y)$ be a function on the unit disc $D = \{(x,y) : x^2 + y^2 < 1\}$ with $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist for all $(x,y) \in D$. Prove or disprove the following each statement.

(a) If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are bounded on D, then f is continuous on D;

(b) If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous on D, then f is differentiable on D.

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9. Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Prove

(a) For any $x, y \in (0, \infty)$

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

(b) If f and g are in $L^p[a,b]$ and $L^q[a,b]$, respectively, then $f(x)g(x)$ is Lebesgue integrable and we have

$$\int_a^b f(x)g(x)dx \leq \Big(\int_a^b |f(x)|^p dx\Big)^{1/p} \Big(\int_a^b |g(x)|^q dx\Big)^{1/q}.$$

Problem 9 (continued)