ALGEBRA QUALIFYING EXAM

Spring 2015

Instructions: LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let $\mathbb{F}_q$ denote the finite field with q elements. Let $\mathbb{Z}$ denote the integers. Let $\mathbb{Q}$ denote the rational numbers.

1. Prove that every finite group of order > 2 has a nontrivial automorphism.

2. In this problem there is no need to justify your answers.

   (a) Define **UFD**.

   (b) Define **PID**.

   (c) For the properties "UFD" and "PID", give an example of a commutative integral domain that

      i. satisfies both properties,

      ii. satisfies one property but not the other,

      iii. satisfies neither property.

3. (a) Prove that $\mathbb{Q}(\sqrt[4]{T})$ is not Galois over $\mathbb{Q}(T)$, where T is an indeterminate.

   (b) Find the Galois closure of $\mathbb{Q}(\sqrt[4]{T})$ over $\mathbb{Q}(T)$ and determine the Galois group both as an abstract group and as a set of explicit automorphisms. (Fully justify.)

4. Let R be a commutative ring with multiplicative identity. An element $r \in R$ is called **nilpotent** if there exists a positive integer n such that $r^n = 0$.

   (a) Prove that every nilpotent element lies in every prime ideal.

   (b) Assume that every element of R is either nilpotent or a unit. Prove that R has a unique prime ideal.

5. For every positive integer n, denote by C_n a cyclic group of order n and by D_n a dihedral group of order $2n$, so that

$$D_n = \{1, a, a^2, \ldots, a^{n-1}, b, ba, ba^2, \ldots, ba^{n-1}\}$$

   where a has order n, b has order 2 and $ab = ba^{-1}$.

   (a) In the notation explained above, prove that every subgroup of $\langle a \rangle$ is normal in D_n.

   (b) If $n = 2m$ with m odd, prove that $D_n = D_{2m} \simeq C_2 \times D_m$.

   (c) Is $D_{12} \simeq C_3 \times D_4$? Justify your answer.

6. Suppose that p and q are prime numbers with $p < q$. Prove that no group of order p^2q is simple.

7. Determine the maximal ideals of the following rings (fully justify):

   (a) $\mathbb{Q}[x]/\langle x^2 - 5x + 6 \rangle$,

   (b) $\mathbb{Q}[x]/\langle x^2 + 4x + 6 \rangle$.

8. Find two matrices having the same characteristic polynomials and minimal polynomials but different Jordan canonical forms. Fully justify.

9. (a) What does it mean for a field to be **perfect**?

   (b) Give an example of a perfect field. (No need to justify your answer.)

   (c) Give an example of a field that is **not** perfect. (No need to justify your answer.)

10. (a) Classify the conjugacy classes of the symmetric group S_3 and justify.

    (b) Construct the character table of S_3.