ALGEBRA QUALIFYING EXAM

Fall 2015

Instructions: LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let $\mathbb{F}_q$ denote the finite field with q elements. Let $\mathbb{Z}$ denote the integers. Let $\mathbb{Q}$ denote the rational numbers. Let $\mathbb{R}$ denote the real numbers. Let $\mathbb{C}$ denote the complex numbers.

1. (a) Define **prime ideal**.
   (b) Define **maximal ideal**.
   (c) Give an example of a ring R and ideals P_1, P_2, and P_3 of R such that for the properties "prime ideal" and "maximal ideal" of R,
      i. P_1 satisfies both properties,
      ii. P_2 satisfies neither property,
      iii. P_3 satisfies one property but not the other.

   Justify your answers.

2. Show that if a group G has only finitely many subgroups then G is a finite group.

3. Let A be an $n \times n$ matrix with entries in $\mathbb{R}$ such that $A^2 = -I$.
   (a) Prove that n is even.
   (b) Prove that A is diagonalizable over $\mathbb{C}$ and describe the corresponding diagonal matrices.

4. Let G be a group of order 70. Prove that G has a normal subgroup of order 35.

5. Construct a Galois extension F of $\mathbb{Q}$ satisfying $\mathrm{Gal}(F/\mathbb{Q}) \simeq D_8$, the dihedral group of order 8. Fully justify.

6. Let F be a field. Prove that every ideal of $F[x]$ is principal.

7. Give an example of a module M over a ring R such that M is **not** finitely generated as an R-module. Prove that it is not finitely generated as an R-module.

8. Suppose H is a normal subgroup of a finite group G.
   (a) Prove or disprove: If H has order 2, then H is a subgroup of the center of G.
   (b) Prove or disprove: If H has order 3, then H is a subgroup of the center of G.

9. (a) What does it mean for a representation to be **irreducible**?
   (b) Suppose p is a prime. Let $G = \mathbb{Z}/p\mathbb{Z}$ and let $\rho : G \to \mathrm{GL}_2(\mathbb{F}_p)$ be a representation. Show that ρ is reducible.

10. (a) Compute the order of $\mathrm{GL}_4(\mathbb{F}_{3^2})$. (Justify your reasoning.)
    (b) Compute the order of $\mathrm{SL}_4(\mathbb{F}_{3^2})$. (Justify your reasoning.)
    (c) Show that $\mathbb{Z}[\sqrt{10}]$ is not a UFD.