ALGEBRA QUALIFYING EXAM

September 22, 2014

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

1. Give definitions for each of the following:
   (a) the group S_n,
   (b) ring,
   (c) ring homomorphism,
   (d) field,
   (e) PID.
2. Let F be a field. Prove that $\langle F, + \rangle$ and $\langle F^\times, \cdot \rangle$ are not isomorphic as groups.
3. True/False. Answer "True" or "False", and fully justify your answer with a proof or counterexample:
   If R is a principal ideal domain and P a nonzero prime ideal of R, then P is a maximal ideal of R.
4. Prove that no group of order 132 is simple.
5. Determine the splitting field over $\mathbb{Q}$ of the polynomial $x^4 + x^2 + 1$, and the degree over $\mathbb{Q}$ of the splitting field.
6. Find two 4×4 matrices with the same characteristic and minimal polynomials that are not similar. (Fully justify your answer.)
7. (a) Let R be an integral domain. Prove that $R[x]^\times = R^\times$, i.e. the units in $R[x]$ are the constant polynomials whose constant term is a unit in R.
   (b) Find an example of a ring R and nonconstant polynomials $f(x), g(x) \in R[x]$ such that $f(x)g(x) = 1$.
8. Let p be a prime. Prove that the Galois group for $x^p - 2$ over $\mathbb{Q}$ is isomorphic to the group of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

   with $a, b \in \mathbb{F}_p$, $a \neq 0$.
9. Determine all real matrices A with characteristic polynomial $x^3(x^2 + 1)$, up to conjugation.
10. Let $\mathbb{F}_q$ denote the finite field with q elements, and let $\mathbb{F}_q^*$ denote the elements of $\mathbb{F}_q$ that have a multiplicative inverse. Suppose p is a prime, and suppose r and N are positive integers. Consider the map $\mathbb{F}_{p^r}^* \to \mathbb{F}_{p^r}^*$ that sends x to x^N. What is the cardinality of its kernel and image? Fully justify.