Algebra Qualifying Examination

September, 2002

The 10 exam Problems: Do as many problems as you can. We prefer complete solutions of a few problems to many partial solutions. 60 total points is sufficient to pass at the Master's Level. 75 total points is sufficient to pass at the PhD level. Do each problem on a separate page. Write your name on each page. If you don't understand some terminology, please ask. The notation throughout a given problem remains constant. Show all details and quote theorems you use properly.

Notations: The set of permutations on n integers is the group S_n. The invertible $n \times n$ matrices over a field K is $\mathrm{GL}_n(K)$. Recall: A permutation representation of G is the action of G on the cosets of some subgroup H of G. The *degree* of the representation is the index of H in G. A permutation representation is faithful if no nonidentity element of G fixes each coset of H.

If a group G acts on a vector space V, we say V is irreducible under the action of G if it fixes no proper subspace of V. For R a ring, R^* is the units of R. As usual $\mathrm{SO}(3, \mathbb{R})$ is the group of 3×3 matrices over $\mathbb{R}$ of determinant $+1$ whose transpose is its inverse ($O(n, \mathbb{R})$ is the orthogonal group - the group of $n \times n$ matrices whose transpose is theirs inverse). Also $\mathrm{U}(2, \mathbb{C}) = \mathrm{U}(2)$ is the group of 2×2 matrices over $\mathbb{C}$ whose transpose conjugate is its inverse. The subgroup of $\mathrm{U}(2)$ consisting of matrices with determinant equal to one is $\mathrm{SU}(2)$.

1 The Operator of Differentiation

4 1.a) Let D be the operator of differentiation on a space of polynomials of degree less or equal to n defined over $\mathbb{C}$. Find its Jordan normal form.

6 1.b) Consider the polynomials of two variables x, y of degree less or equal to $n-1$ defined over $\mathbb{C}$. Let $D = \partial/\partial x + \partial/\partial y$. Find its Jordan normal form.

2 The Alternating Group A_5

4 2.a) Describe the conjugacy classes of A_5.

32.b) Show that if G is a group and H is a normal subgroup in G then H is an union of conjugacy classes.

3 2.c) Using the a) and b) show that the group A_5 is simple.

3 Abelian groups

Suppose p_1 and p_2 are two primes with $p_1 < p_2$.

3 **3.a)** What are the possible abelian groups of order $p_1 p_2$? Explain in detail why all abelian groups of order $p_1 p_2$ are isomorphic.

3 **3.b)** Show that if a group G is of order $p_1 p_2$ and p_1 does not divide $p_2 - 1$, then G is abelian.

3 **3.c)** Describe all abelian groups of order 64 up to an isomorphism.

3 **3.d)** Is it true that all groups of order n^3 are abelian? Please explain.

4 Matrices

4 **4.a)** Let A and B be any $n \times n$ matrices over $\mathbb{C}$. Is it possible that $AB - BA = E$? Please explain. Here E is the identity matrix.

6 **4.b)** Let A and B be any $n \times n$ matrices over $\mathbb{Z}/p\mathbb{Z}$ for some $p \geq 2$. Is it possible that $AB - BA = E$? Please explain.

5 The Group $SO(3, \mathbb{R})$

This problem outlines a proof that $SO(3, \mathbb{R})$ is a simple group. Use without a proof that there exists a surjective homomorphism $\rho : SU(2) \to SO(3, \mathbb{R})$ such that $\ker(\rho) = \mathbb{Z}/(2)$.

3 **5.a)** Let G be any group and let H be any normal subgroup of G. Show that if $a \in G$ and $h \in H$, then $[h, a] = hah^{-1}a^{-1} \in H$.

4 **5.b)** Describe all conjugacy classes of $SU(2)$. Hint: Use Jordan canonical form for 2×2 matrices over $\mathbb{C}$.

4 **5.c)** Use a) and b) to show $SU(2)$ has only one nontrivial normal subgroup and derive from there that $SO(3, \mathbb{R})$ is a simple group.

6 Multiplicative Group of a Finite Field

Let p be a prime. Suppose $a \in \mathbb{Z}$ is a product of distinct primes $q_1 < \cdots < q_r$. Use without proof that there is an $\alpha \in \mathbb{F}_p \setminus \{0\} = \mathbb{F}_p^*$ that generates $\mathbb{F}_p^*$.

4 **6.a)** Show that $x^2 = q_i \mod p$ has a solution in $\mathbb{F}_p$ if and only if $q_i = \alpha^{2k}$ for some integer k.

3 **6.b)** Suppose $q_i \equiv \alpha^{k_i} \mod p$. Show that $x^2 \equiv a \mod p$ has a solution in $\mathbb{F}_p$ if and only if $\sum_{i=1}^{r} k_i$ is even.

4 6.c) Let P_q be the set of primes not in the set $\{q_1, \ldots, q_r\}$. For each $p \in P_q$, let $v_p \in \mathbb{Z}_2^r = U$ be $(\epsilon_1, \ldots, \epsilon_r)$ where $\epsilon_i = 0, 1$ if $x^2 \equiv q_i \mod p$ has (resp. does not have) a solution. Suppose $p_1, p_2, p_3 \in P_q$ are three primes satisfying $p_1 p_2 \equiv p_3 \mod q_i$. Show that $v_{p_1} + v_{p_2} = v_{p_3}$.

7 Applications of Chevalley's Theorem

Let $F(x_1, ..., x_n)$ is a homogeneous polynomial of degree $r < n$. Then Chevalley's theorem states that the number of solutions of this polynomial over $\mathbb{Z}/p\mathbb{Z}$ is divisible by p.

4 7.a) Show that condition $r < n$ in Chevalley's theorem is an essential condition. Hint: Consider polynomial $x^2 - 2y^2$ over $\mathbb{Z}/5\mathbb{Z}$.

6 7.b) Consider the circle $x^2 + y^2 = 1$ over $\mathbb{Z}/p\mathbb{Z}$, where p is a prime number, $p > 5$. Then there is a point with coordinates $x, y \in \mathbb{Z}/p\mathbb{Z}$ on the circle. Hint; Consider equation $x^2 + y^2 - z^2 = 0$ and use Chevalley's theorem.

8 The Galois Group of Degree 5 Polynomial

5 8.a) Let $f(x)$ be an irreducible degree five polynomial over $\mathbb{Q}$ with exactly 3 real roots. Show that the Galois group of $f(x)$ is S_5.

3 8.b) Consider the polynomial $f(x) = x^5 - 25x + 3$. Show that it is irreducible.

3 8.c) Using a) and b) find the Galois group of $f(x) = x^5 - 25x + 3$.

9 Representations of Q_8 - the Quaternions Group of Order 8.

This problem describes all irreducible representations of Q_8. Recall: The number of irreducible representations of a finite group G is the same as the number of conjugacy classes in G. Further, the sums of squares of the degrees of these representations is $|G|$.

3 9.a) Give a representative of each conjugacy class in Q_8. Find the center and the commutant of Q_8.

4 9.b) Show that every subgroup of Q_8 is normal.

3 9.c) List all irreducible representations of Q_8.