REAL ANALYSIS QUALIFYING EXAMINATION, JUNE 15, 2011

10-12:30, RH 114

Last name:____________________ First name:____________________

UCI ID:____________________ Signature:____________________

Instructions:

(1) Solve as many problems as you can.
(2) Use only one side of each sheet. Do at most one problem on problem on each page.
(3) Write your name, and the number of the problem you are working on, on every page.
(4) Justify your answers. Where appropriate, state without proof the results you are using.
(5) Throughout the test, standard notation is used. For instance, μ_L stands for the usual Lebesgue measure on $\mathbb{R}$.

Good luck!

Problem	Max. grade	Your grade
1	10 pts.	
2	10 pts.	
3	10 pts.	
4	10 pts.	
5	10 pts.	
6	10 pts.	
Total:	60 pts.	

Throughout the test, standard notation is used. For instance, μ_L stands for the usual Lebesgue measure on $\mathbb{R}$.

Problem 1. Suppose f and g are real-valued μ_L-measurable functions on $\mathbb{R}$, such that (1) f is μ_L-integrable, and (2) g belongs to $C_0(\mathbb{R})$. For $c > 0$ define $g_c(t) = g(ct)$. Prove that (a) $\lim_{c\to\infty} \int_{\mathbb{R}} f g_c \, d\mu_L = 0$, and (b) $\lim_{c\to 0} \int_{\mathbb{R}} f g_c \, d\mu_L = g(0) \int_{\mathbb{R}} f \, d\mu_L$.

Problem 2. Suppose μ and ν are σ-finite measures on a measurable space $(X, \mathcal{A})$, such that $\nu \le \mu$, and $\nu \ll \mu - \nu$. Prove that

$$\mu\Big(\Big\{x \in X : \frac{d\nu}{d\mu} = 1\Big\}\Big) = 0.$$

Problem 3. Prove that the Gamma function

$$\Gamma(x) = \int_{(0,\infty)} t^{x-1} \mathrm{e}^{-t} \, \mu_L(dt)$$

is well defined and continuous for $x > 0$.

Problem 4. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be the measure spaces given by:

- $X = Y = [0, 1]$.
- $\mathcal{A} = \mathcal{B} = \mathcal{B}_{[0,1]}$, the Borel σ-algebra of $[0, 1]$.
- $\mu = \mu_L$, and ν is the counting measure.

Consider the product measurable space $(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}))$, and its subset $E = \{(x, y) \in X \times Y;\ x = y\} \subset X \times Y$.

(1) Show that $E \in \sigma(\mathcal{A} \times \mathcal{B})$.
(2) Show that $\int_X \big\{\int_Y \mathbf{1}_E \, d\nu\big\} \, d\mu \neq \int_Y \big\{\int_X \mathbf{1}_E \, d\mu\big\} \, d\nu$.
(3) Explain why Tonelli's Theorem is not applicable.

Problem 5. Suppose $f \in C^1[0, 1]$ (that is, f is continuous, and continuously differentiable, on $[0, 1]$), $f(0) = f(1)$, and $f > f'$ everywhere.

(1) Prove that $f > 0$ everywhere.

(2) Prove that

$$\int_{(0,1)} \frac{f^2}{f - f'} d\mu_L \ge \int_{(0,1)} f d\mu_L.$$

Hint. Apply Cauchy-Schwarz (or Hölder) Inequality to the product of h and g, where $h = \sqrt{f - f'}$, and $g = f/h$.

Problem 6. Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of measurable functions on $[0, 1]$. For $x \in [0, 1]$ define $h(x) = \#\{n : f_n(x) = 0\}$ (the number of indices n for which $f_n(x) = 0$). Assuming that $h < \infty$ everywhere, prove that the function h is measurable.