Instructions *Complete all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

1. For each of (a) and (b) either prove or give a counterexample. Suppose that for every $n \in \mathbb{N}$, $f_n : X \to [0, 1]$ is a μ-measurable function, and
$$\lim_{n\to\infty} \int f_n \, d\mu = 0.$$
Then
   (a) $f_n \to 0$ in measure
   (b) for μ-almost all $x \in X$, $\lim_{n\to\infty} f_n(x) = 0$.

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be Borel measurable for every $n \in \mathbb{N}$. Define E to be the set of points in $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x)$ exists and is finite. Show that E is a Borel measurable set.

3. Does there exist a nowhere dense subset of $[0, 1]^2 \subset \mathbb{R}^2$
   (a) of Lebesgue measure greater than 9/10?
   (b) of Lebesgue measure 1 ?

   If yes, construct such a set, if no, prove why not.

4. Let $\lambda \in (0, 1)$, and $f \in L^1([0, 1])$. Show that the integral
$$F(x) = \int_0^x \frac{1}{(x-t)^\lambda} f(t) \, dt$$
exists a.e. $x \in [0, 1]$ and that $F \in L^1([0, 1])$.

5. Show that for $g \in L^3([0, 1])$, $f \in L^{3/2}([0, 1])$
$$\int_0^1 \cos 2\pi nx \, g(x)f(x)dx \to 0$$
as $n \to \infty$.

6. Assume that f, $f' \in L^1(\mathbb{R})$. Let
$$g(x) = \sum_{k=0}^{\infty} |f(x + k)|.$$
Show that $g \in L^\infty(I)$ for any bounded interval I. (Hint: first prove for intervals with length $\leq \frac{1}{2}$).