Qualifying Exam in Real Analysis, Fall Quarter 2012

Instructions *Complete all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

1. Find

   (a) $\lim_{n\to\infty} \int_0^\infty e^{-\frac{x^2+3x}{n^2}-2x} dx$

   (b) $\lim_{n\to\infty} \int_0^\pi x^2 \cos \frac{3x}{n} dx.$

   Justify your answers.

2. Find a sequence of pointwise convergent measurable functions $f_n : [0,1] \to \mathbb{R}$ such that f_n doesn't converge uniformly on any set X with $m(X) = 1$.

3. Suppose f is Borel measurable and
$$\int_0^1 x^{-1/2} |f(x)|^3 dx < \infty.$$
   Show that $\lim_{t\to 0} t^{-5/6} \int_0^t f(x)dx = 0.$

4. If (X, μ) is a finite measure space, $1 < p_1 < p_2 < \infty$, $f \geq 0$, and
$$\sup_{\lambda>0} \lambda^{p_2} \mu(x : \ f(x) > \lambda) < \infty,$$
   prove that $f \in L^{p_1}(\mu)$.

5. Let $x = 0.n_1 n_2 \ldots$ be a decimal representation of $x \in [0,1]$. Let $f(x) = \min_i n_i$. Prove that $f(x)$ is measurable and a.e. constant.

6. Let f_n be a sequence of measurable functions on (X, μ) with $f_n \geq 0$ and $\int_X f_n d\mu = 1$. Show that
$$\limsup_{n\to+\infty} f_n^{\frac{1}{n}}(x) \leq 1 \quad \text{for } \mu \text{ a.e. } x.$$