Print Your Math Exam Id: ______________

Complex Qualifying Examination

Time: 1:00 pm–3:30 pm, 9/21/2017
Room: MSTB 124

Table of your scores

Problem 1	________/ 10
Problem 2	________/ 10
Problem 3	________/ 10
Problem 4	________/ 10
Problem 5	________/ 10
Problem 6	________/ 10
Problem 7	________/ 10
Problem 8	________/ 10
Total	________/ 80

Notation:

C denotes the complex plane; $i = \sqrt{-1}$;
$D(z_0, r)$ denotes the open disc in **C** centered at z_0 and radius r.

1. Let u be a real-valued continuous function on $\mathbf{C}$ such that $e^{u(z)}$ is harmonic in $\mathbf{C}$. Then u is a constant.

2. Prove or disprove there is a holomorphic function f on the unit disk $D(0,1)$ such that

$$f(\frac{1}{2n}) = f(\frac{1}{2n+1}) = \frac{1}{n}$$

for all positive integers n.

3. Let f be an entire holomorphic function in $\mathbf{C}$ such that $f(x)$ and $f(ix)$ are real for $x \in (1, 2)$. Prove there is an entire function g such that

$$f(z) = g(z^2), \quad z \in \mathbf{C}.$$

4. Let $z_1, \cdots, z_n \in D(0, R)$ and

$$Q(z) = (z - z_1) \cdots (z - z_n)$$

Let f be a holomorphic function on $\overline{D(0, R)}$. Prove

$$P(z) = \frac{1}{2\pi i} \int_{|w|=R} f(w) \frac{Q(w) - Q(z)}{(w - z)Q(w)} dw$$

is a polynomial of degree $n - 1$ such that $f(z_j) = P(z_j)$ for $1 \leq j \leq n$.

5. For $a > 1$ Prove the equation $ze^{a-z} = 1$ has a unique solution in $|z| \leq 1$, which is also real and positive.

6. Prove

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8}$$

7. Let $\mathcal{F}$ be a family of holomorphic functions f on the unit disc $D(0,1)$ such that

$$|f(0)|^2 + \int_{D(0,1)} |f'(z)|^2 dA \leq 1.$$

Prove that $\mathcal{F}$ is a normal family.

8. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on the unit disk $D(0,1)$ such that

$$F(z) = \sum_{n=1}^{\infty} |f_n(z)|$$

defines a continuous function in $D(0,1)$ and $F(0) \geq F(z)$ on $D(0,1)$. Prove f_n are constant for all $n = 1, 2, 3, ...$